Quantum Koopmanism

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The Koopman-von Neumann spectral approach to ergodic theory is a powerful tool in the study of statistical properties of dynamical systems (see [Ergodic theory. Mixing], [Spectrum of dynamical system]). Its extension to quantum dynamical systems – the spectral theory of Liouvilleans – is at the center of many recent results in quantum statistical mechanics (see [Quantum nonequilibrium statistical mechanics], [Return to equilibrium], [NESS in quantum statistical mechanics] as well as [BFS],[DJ], [FM], [JP]).

Let $(\mathcal{O},\tau)$ be a C^*- or W^*-dynamical system equipped with a τ-invariant state ω, assumed to be normal in the W^*-case. The GNS-representation $(\mathcal{H}_\omega,\pi_\omega,\Omega_\omega)$ maps the triple $(\mathcal{O},\tau,\omega)$ into $(\mathcal{O}_\omega,\tilde\tau,\tilde\omega)$, a W^*-dynamical system on the enveloping von Neumann algebra $\mathcal{O}_\omega=\pi_\omega(\mathcal{O})''$ with a normal invariant state $\tilde\omega(A)=(\Omega_\omega|A\Omega_\omega)$. The W^*-dynamics $\tilde\tau$ is given by

$$\tilde\tau^t(A)=\mathrm{e}^{\mathrm{i}tL_\omega}A\mathrm{e}^{-\mathrm{i}tL_\omega},$$

where L_ω is the ω-Liouvillean (see Section 3 in [Quantum dynamical systems]).

We shall say that $(\pi_\omega,\mathcal{O}_\omega,\mathcal{H}_\omega,L_\omega,\Omega_\omega)$ is the normal form of $(\mathcal{O},\tau,\omega)$.

1 Ergodic properties of quantum dynamical systems

Let $\mathfrak{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$. The support s_ω of a normal state ω on $\mathfrak{M}$ is the smallest orthogonal projection $P\in\mathfrak{M}$ such that $\omega(P)=1$. A normal state ω is faithful if and only if $s_\omega=I$. The support of the state $\omega(A)=(\Omega|A\Omega)$ is the orthogonal projection on the closure of the subspace $\mathfrak{M}'\Omega$.

Notation. We write $\nu\ll\omega$ whenever ν is a ω-normal state such that $s_\nu\le s_\omega$.

Remark. If $\mathfrak{M}$ is Abelian any ω-normal state ν satisfies $\nu\ll\omega$. This explains why the support condition is absent in classical ergodic theory (the reader may consult [P] for a detailed discussion of this point). In most applications to statistical mechanics ω is faithful and any ω-normal state ν satisfies $\nu\ll\omega$.

Definition 1 *Let $(\mathfrak{M},\tau)$ be a W^*-dynamical system on the von Neumann algebra $\mathfrak{M}$ and ω a normal τ-invariant state.*

1. *$(\mathfrak{M},\tau,\omega)$ is ergodic if*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\nu(\tau^t(A))\,\mathrm{d}t=\omega(A),$$

holds for all $A\in\mathfrak{M}$ and all states $\nu\ll\omega$.

2. *$(\mathfrak{M},\tau,\omega)$ is mixing or returns to equilibrium if*

$$\lim_{t\to\infty}\nu(\tau^t(A))=\omega(A),$$

holds for all $A\in\mathfrak{M}$ and all states $\nu\ll\omega$.

3. *If ω is an invariant state of the C^*-dynamical system $(\mathcal{O},\tau)$ we say that $(\mathcal{O},\tau,\omega)$ is ergodic (resp. mixing) if $(\mathcal{O}_\omega,\tilde\tau,\tilde\omega)$ is ergodic (resp. mixing).*

2 Spectral characterization of ergodic properties

We refer to [P] for proofs of the results in this section.

The following theorem is the quantum version of the well known Koopman-von Neumann spectral characterizations ([AA], [K], [N]).

Theorem 2 *Let $(\mathcal{O}, \tau)$ be a C^*- or W^*-dynamical system equipped with a τ-invariant state ω, assumed to be normal in the W^*-case. Denote by $(\pi_\omega, \mathcal{O}_\omega, \mathcal{H}_\omega, L_\omega, \Omega_\omega)$ its normal form and by $\mathcal{K}_\omega$ the closure of $\pi_\omega(\mathcal{O})'\Omega_\omega$.*

1. *The subspace $\mathcal{K}_\omega$ reduces the operator L_ω. Denote by $\mathfrak{L}_\omega$ the restriction $L_\omega\big|_{\mathcal{K}_\omega}$.*
2. *$(\mathcal{O}, \tau, \omega)$ is ergodic if and only if $\mathrm{Ker}(\mathfrak{L}_\omega)$ is one dimensional.*
3. *$(\mathcal{O}, \tau, \omega)$ is mixing if and only if*
$$\mathrm{w}-\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}t\mathfrak{L}_\omega} = \Omega_\omega(\Omega_\omega|\cdot).$$
4. *If the spectrum of $\mathfrak{L}_\omega$ on $\{\Omega_\omega\}^\perp$ is purely absolutely continuous then $(\mathcal{O}, \tau, \omega)$ is mixing.*

Note that $\mathcal{K}_\omega$ is the range of the support of $\tilde{\omega}$. Thus, if $\tilde{\omega}$ is faithful then $\mathfrak{L}_\omega = L_\omega$.

Like the classical Koopman operator, the reduced Liouvillean $\mathfrak{L}_\omega$ of an ergodic quantum dynamical system has a number of peculiar spectral properties.

Theorem 3 *Assume, in addition to the hypotheses of the previous theorem, that $(\mathcal{O}, \tau, \omega)$ is ergodic. Then the following hold:*

1. *The point spectrum of $\mathfrak{L}_\omega$ is a subgroup Σ of the additive group $\mathbb{R}$.*
2. *The eigenvalues of $\mathfrak{L}_\omega$ are simple.*
3. *The spectrum of $\mathfrak{L}_\omega$ is invariant under translations in Σ, that is, $\mathrm{spec}(\mathfrak{L}_\omega) + \Sigma = \mathrm{spec}(\mathfrak{L}_\omega)$.*
4. *If Ψ is a normalized eigenvector of $\mathfrak{L}_\omega$ then $(\Psi|\pi_\omega(A)\Psi) = \omega(A)$ for all $A \in \mathcal{O}$.*
5. *If $(\mathcal{O}, \tau, \omega)$ is mixing then 0 is the only eigenvalue of $\mathfrak{L}_\omega$.*

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