On entropy production in quantum statistical mechanics

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Abstract

We propose a definition of entropy production in the framework of algebraic quantum statistical mechanics. We relate our definition to heat flows through the system. We also prove that entropy production is non-negative in natural nonequilibrium steady states.

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1 Introduction

Let $(\mathcal{O},\tau)$ be a C^*-dynamical system, where $\mathcal{O}$ is a C^*-algebra with identity and τ a strongly continuous group of automorphisms of $\mathcal{O}$ (strong continuity means that the map $\mathbf{R} \ni t \mapsto \tau^t(A)$ is continuous in norm for each $A \in \mathcal{O}$). The elements of $\mathcal{O}$ describe observables of the physical system under consideration. The group τ specifies their time evolution. A physical state of the system is described by a mathematical state on $\mathcal{O}$, that is, a positive linear functional ω such that $\omega(\mathbf{1}) = 1$. The set $E(\mathcal{O})$ of all states on $\mathcal{O}$ is a convex, weak-$*$ compact subset of the dual $\mathcal{O}^*$. A state ω is invariant under the group τ if $\omega \circ \tau^t = \omega$ for all t. For our purposes we assume that in addition to $(\mathcal{O},\tau)$ we are given a τ-invariant state ω.

The triple $(\mathcal{O},\tau,\omega)$ describes a physical system in a steady state. We are interested in effects of local perturbations on such system. A local perturbation is specified by a self-adjoint element V of $\mathcal{O}$ and in what follows we fix such a V. The perturbed time evolution is given by

$$\tau_V^t(A) \equiv \tau^t(A) + \sum_{n\geq 1} \mathrm{i}^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n [\tau^{t_n}(V), [\ldots [\tau^{t_1}(V), A]]].$$

The pair $(\mathcal{O},\tau_V)$ is also a C^*-dynamical system. Following Ruelle [Ru2], we call the weak-$*$ limit points of the set

$$\left\{ \frac{1}{T} \int_0^T \omega \circ \tau_V^t \, dt \,\middle|\, T > 0 \right\} \subset E(\mathcal{O}),$$

nonequilibrium steady states (NESS) of the locally perturbed system. The set $\Sigma_V^+(\omega)$ of NESS of $(\mathcal{O},\tau_V)$ is a non-empty, compact subset of $E(\mathcal{O})$ whose elements are τ_V-invariant.

Our first assumption is:

(A1) There exists a strongly continuous group σ_ω of automorphims of $\mathcal{O}$ such that ω is $(\sigma_\omega, -1)$-KMS state.

Let δ_ω be the generator of σ_ω (i.e. $\sigma_\omega^t = \mathrm{e}^{t\delta_\omega}$). We denote by $\mathcal{D}(\delta_\omega)$ the domain of δ_ω. $\mathcal{D}(\delta_\omega)$ is a norm-dense $*$-subalgebra of $\mathcal{O}$ and for $A, B \in \mathcal{D}(\delta_\omega)$,

$$\delta_\omega(A)^* = \delta_\omega(A^*), \qquad \delta_\omega(AB) = \delta_\omega(A)B + A\delta_\omega(B).$$

Our second assumption is:

(A2) $V \in \mathcal{D}(\delta_\omega)$.

We define the observable

$$\sigma_V \equiv \delta_\omega(V),$$

and for reasons which will soon become clear, we call

$$\mathrm{Ep}_V(\eta) \equiv \eta(\sigma_V),$$

the entropy production (with respect to the reference state ω) of the perturbed system $(\mathcal{O}, \tau_V)$ in the state $\eta \in E(\mathcal{O})$. Note that σ_V, and hence $\mathrm{Ep}_V(\cdot)$, depend in a non-trivial way on the state ω.

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation of the algebra $\mathcal{O}$ associated to ω, and let $\mathcal{N}_\omega$ be the set of π_ω-normal states on $\mathcal{O}$, that is, the states represented by density matrices on $\mathcal{H}_\omega$. Any $\eta \in \mathcal{N}_\omega$ has a continuous extension to $\mathfrak{M} \equiv \pi_\omega(\mathcal{O})''$ which we denote by the same letter. For $\eta, \xi \in \mathcal{N}_\omega$, we denote by $\mathrm{Ent}(\eta \,|\, \xi)$ the relative entropy of η with respect to ξ. (We use the definition of relative entropy given in [BR2], Definition 6.2.29. This definition differs by a sign and the order of factors from the original Araki's definition [Ar].)

Our main result, which justifies the above definition of entropy production, is:

Theorem 1.1 *Assume that* (A1) *and* (A2) *hold. Then, for any faithful state* $\eta \in \mathcal{N}_\omega$ *such that* $\mathrm{Ent}(\eta \,|\, \omega) > -\infty$*, one has*

$$\mathrm{Ent}(\eta \circ \tau_V^t \,|\, \omega) = \mathrm{Ent}(\eta \,|\, \omega) - \int_0^t \mathrm{Ep}_V(\eta \circ \tau_V^s)\, ds.$$

Remark. The same result (with the same proof) holds for W^*-dynamical systems.

In the rest of this section we will discuss some elementary properties of $\mathrm{Ep}_V(\cdot)$. Let $\omega_V^+ \in \Sigma_V^+(\omega)$ and $T_n \to \infty$ be such that

$$\lim_n \frac{1}{T_n} \int_0^{T_n} \omega \circ \tau_V^t \, dt = \omega_V^+. \tag{1.1}$$

Then, with the particular choice $\eta = \omega$, Theorem 1.1 gives

$$\lim_n \frac{1}{T_n} \mathrm{Ent}(\omega \circ \tau_V^{T_n} \,|\, \omega) = -\lim_n \frac{1}{T_n} \int_0^{T_n} \omega(\tau_V^s(\sigma_V))\, ds = -\mathrm{Ep}_V(\omega_V^+). \tag{1.2}$$

Since the relative entropy is non-positive, we immediately get

Theorem 1.2 *Assume that* (A1) *and* (A2) *hold. Then, for any* NESS $\omega_V^+ \in \Sigma_V^+(\omega)$*, one has*

$$\mathrm{Ep}_V(\omega_V^+) \geq 0.$$

With regard to (1.2), on physical grounds one expects that the ratio

$$\mathrm{Ent}(\eta \circ \tau_V^t \,|\, \omega)/t$$

becomes independent of the choice of the reference state ω as $t \to \infty$. More precisely, the following result holds:

Proposition 1.3 *Assume* (A1) *and that* $\eta \in \mathcal{N}_\omega$ *is faithful. Then there is a norm-dense set* $\mathcal{N}'_\omega \subset \mathcal{N}_\omega$ *such that for* $\omega' \in \mathcal{N}'_\omega$*,*

$$\mathrm{Ent}(\eta \circ \tau_V^t \,|\, \omega') = \mathrm{Ent}(\eta \circ \tau_V^t \,|\, \omega) + O(1),$$

as $t \to \infty$*.*

One also expects that in thermal equilibrium the entropy production is zero, that is, if $\eta \in \mathcal{N}_\omega$ is a (τ_V, β)-KMS state then $\mathrm{Ep}_V(\eta) = 0$. In fact, a much stronger result holds.

Proposition 1.4 *Assume* (A1), (A2) *and that* $\eta \in \mathcal{N}_\omega$ *is a faithful,* τ_V*-invariant state. Then*

$$\mathrm{Ep}_V(\eta) = 0.$$

Remark. Again, this result also holds for W^*-dynamical systems.

Let $\mathcal{O}$ be the CAR algebra over $l^2(\mathbf{Z}^3)$ describing a free Fermi gas on the lattice $\mathbf{Z}^3$. Using some technical results proven in [BM] it is easy to construct a large class of quasi-free states ω and local perturbations V such that (A1)-(A2) hold, and that $\Sigma_V^+(\omega)$ consists of a single state ω_V^+. In these examples, $\mathrm{Ep}_V(\omega_V^+)$ can be computed perturbatively (similar calculations are done in [HTP]), and one easily constructs examples where $\mathrm{Ep}_V(\omega_V^+) > 0$.

In the next example we relate entropy production to heat flows.

Consider two independent systems $(\mathcal{O}_i, \tau_i, \omega_i)$, $i = 1, 2$, each of which is in thermal equilibrium at temperature T_i. This means that ω_i is a (τ_i, β_i)-KMS state on $\mathcal{O}_i$ where $\beta_i = 1/T_i$. Let

$$\mathcal{O} = \mathcal{O}_1 \otimes \mathcal{O}_2, \qquad \tau = \tau_1 \otimes \tau_2, \qquad \omega = \omega_1 \otimes \omega_2.$$

($\otimes$ is the C^*-tensor product, see Section 2.7.2 in [BR1]). Let δ be the generator of τ and δ_i the generator of τ_i. Obviously, $\delta = \delta_1 + \delta_2$ (here we write δ_1 for $\delta_1 \otimes \mathbf{1}$, etc). Let $V \in \mathcal{O}$ be such that $V \in \mathcal{D}(\delta_i)$. Then

$$\omega \circ \tau_V^t(V) - \omega(V) = \int_0^t \omega \circ \tau_V^s(\Phi) ds,$$

where Φ is defined by

$$\tau_V^t(\Phi) = \frac{d}{dt}\tau_V^t(V).$$

Obviously, $\Phi = \Phi_1 + \Phi_2$ where $\Phi_i \equiv \delta_i(V)$ describes the energy flux out of the i-th system. Since the states ω_i are KMS, (A1) holds with $\delta_\omega = \delta_{\omega_1} + \delta_{\omega_2}$ and $\delta_{\omega_i} = -\beta_i \delta_i$. Therefore, (A1) and (A2) hold and

$$\beta_1 \Phi_1 + \beta_2 \Phi_2 = -\sigma_V.$$

It follows that in a NESS $\omega_V^+ \in \Sigma_V^+(\omega)$, the energy fluxes satisfy

$$\frac{\omega_V^+(\Phi_1)}{T_1} + \frac{\omega_V^+(\Phi_2)}{T_2} = -\mathrm{Ep}_V(\omega_V^+) \le 0.$$

Since $\omega_V^+(\Phi_1) + \omega_V^+(\Phi_2) = 0$, if $T_1 > T_2$, then $\Phi_1 \geq 0$ and the heat flows from the hot to the cold reservoir. This calculation is easily generalized to the case where N-level quantum system is coupled to several independent thermal resevoirs.

We finish this section with the following remarks.

In [JP1] we prove an analog of Theorem 1.1 for time-dependent perturbations and discuss the relation between entropy production and the second law of thermodynamics.

In the forthcoming paper [JP2], we will study NESS, entropy production and heat flows for a model of an N-level quantum system coupled to several independent free Fermi gas reservoirs (similar models have been studied in [D, Ru1]).

The entropy production for quantum spin systems has been studied in the recent preprint [Ru2].

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2 Proofs

We assume that the reader is familiar with the basic results of Tomita-Takesaki modular theory as discussed, for example, in [BR1, BR2, H, OP]. We begin by setting the notation and recalling some well-known facts.

$(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ denotes the GNS representation of the algebra $\mathcal{O}$ associated to ω. By (A1), the vector Ω_ω is cyclic and separating for $\mathfrak{M} \equiv \pi_\omega(\mathcal{O})''$. Moreover, (A1) implies that π_ω is injective. We respectively denote by Δ_ω, J and $\mathcal{P}$ the modular operator, the modular conjugation and the natural cone canonically associated to the pair $(\mathfrak{M}, \Omega_\omega)$. We also adopt the shorthands $\mathcal{L}_\omega = \log \Delta_\omega$ and $j(A) = JAJ$. Note that

$$\pi_\omega(\sigma_\omega^t(A)) = \mathrm{e}^{\mathrm{i}t\mathcal{L}_\omega}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}t\mathcal{L}_\omega}.$$

With a slight abuse of notation, we write $\sigma_\omega^t(A) = \mathrm{e}^{\mathrm{i}t\mathcal{L}_\omega}A\mathrm{e}^{-\mathrm{i}t\mathcal{L}_\omega}$ for $A \in \mathfrak{M}$. By the Tomita-Takesaki theorem, $\sigma_\omega^t(\mathfrak{M}) = \mathfrak{M}$, $j(\mathfrak{M}) = \mathfrak{M}'$ and

$$\mathrm{e}^{\mathrm{i}t\mathcal{L}_\omega}J = J\mathrm{e}^{\mathrm{i}t\mathcal{L}_\omega}. \tag{2.3}$$

The Liouvillean L of the system $(\mathcal{O}, \tau, \omega)$ is the unique self-adjoint operator on $\mathcal{H}_\omega$ such that

$$\begin{aligned} \pi_\omega(\tau^t(A)) &= \mathrm{e}^{\mathrm{i}tL}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL}, \\ L\Omega_\omega &= 0, \end{aligned} \tag{2.4}$$

and one easily shows that

$$\begin{aligned} \mathrm{e}^{\mathrm{i}tL}\mathcal{L}_\omega &= \mathcal{L}_\omega\mathrm{e}^{\mathrm{i}tL} \\ \mathrm{e}^{\mathrm{i}tL}J &= J\mathrm{e}^{\mathrm{i}tL}. \end{aligned} \tag{2.5}$$

The self-adjoint operator

$$L_V \equiv L + \pi_\omega(V) - j(\pi_\omega(V)),$$

is uniquely specified by the following two requirements:

$$\begin{aligned} &\pi_\omega(\tau_V^t(A)) = \mathrm{e}^{\mathrm{i}tL_V}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL_V} \\ &\mathrm{e}^{-\mathrm{i}tL_V}\mathcal{P} \subset \mathcal{P}. \end{aligned} \tag{2.6}$$

The dynamical groups τ and τ_V have natural extensions to $\mathfrak{M}$ for which we use the same notation. Note also that $JL_V + L_V J = 0$, and therefore

$$\mathrm{e}^{\mathrm{i}tL_V}J = J\mathrm{e}^{\mathrm{i}tL_V}. \tag{2.7}$$

A state $\eta \in \mathcal{N}_\omega$ has a unique vector representative $\Omega_\eta \in \mathcal{P}$. Relations (2.6) yield that

$$\Omega_{\eta\circ\tau_V^t} = \mathrm{e}^{-\mathrm{i}tL_V}\Omega_\eta. \tag{2.8}$$

The relative entropy of two faithful states $\eta, \xi \in \mathcal{N}_\omega$ is defined as

$$\mathrm{Ent}(\eta\,|\,\xi) \equiv (\Omega_\eta|\log\Delta_{\xi|\eta}\Omega_\eta),$$

where $\Delta_{\xi|\eta}$ is the relative modular operator. Relative entropy is more conveniently expressed in terms of the Radon-Nikodym cocycle $[D\xi : D\eta]^s$ as

$$\mathrm{Ent}(\eta\,|\,\xi) = \lim_{s\downarrow 0}\frac{\eta([D\xi : D\eta]^s - \mathbf{1})}{\mathrm{i}s}. \tag{2.9}$$

Proof of Theorem 1.1. Let us denote by

$$U(t) \equiv \mathrm{e}^{-\mathrm{i}tL}\mathrm{e}^{\mathrm{i}t(L+\pi_\omega(V))},$$

the propagator in the interaction representation. $U(t)$ is the unique solution of

$$\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}U(t) = \pi_\omega(\tau^{-t}(V))U(t),$$

with initial data $U(0) = \mathbf{1}$. It has a norm convergent Dyson expansion

$$U(t) = \mathbf{1} + \sum_{n=1}^{\infty}\mathrm{i}^n\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\pi_\omega(\tau^{-t_1}(V))\cdots\pi_\omega(\tau^{-t_n}(V)),$$

from which we conclude that $U(t) \in \pi_\omega(\mathcal{O})$. With a slight abuse of notation, we will write δ_ω for $\pi_\omega \circ \delta_\omega$. A simple calculation shows that $U(t) \in \mathcal{D}(\delta_\omega)$ and

$$U^*(t)\delta_\omega(U(t)) = -\delta_\omega(U^*(t))U(t) = \mathrm{i}\int_0^t \pi_\omega(\tau_V^{-s}(\sigma_V))ds. \tag{2.10}$$

We claim that

$$\mathrm{e}^{-\mathrm{i}tL}\mathrm{e}^{\mathrm{i}tL_V} = U(t)j(U(t)). \tag{2.11}$$

To prove this fact, note that after differentiation with respect to t both sides satisfy the same differential equation with the same initial condition at $t=0$.

To compute the relative entropy $\mathrm{Ent}(\eta\circ\tau_V^t\,|\,\omega)$, we will use Equation (2.9) and the fact that the Radon-Nikodym cocycle can be expressed as

$$[D\omega : D\eta\circ\tau_V^t]^s = \Delta^{\mathrm{i}s}_{\omega|\omega\circ\tau_V^t}\Delta^{-\mathrm{i}s}_{\eta\circ\tau_V^t|\omega\circ\tau_V^t}. \tag{2.12}$$

By definition of the relative modular operator, for any $A\in\mathfrak{M}$ we have

$$J\Delta^{1/2}_{\eta\circ\tau_V^t|\omega\circ\tau_V^t}A\Omega_{\omega\circ\tau_V^t} = A^*\Omega_{\eta\circ\tau_V^t}.$$

Using Relations (2.6) and (2.8), we further obtain

$$J\Delta^{1/2}_{\eta\circ\tau_V^t|\omega\circ\tau_V^t}A\mathrm{e}^{-\mathrm{i}tL_V}\Omega_\omega = A^*\mathrm{e}^{-\mathrm{i}tL_V}\Omega_\eta.$$

It follows that

$$\begin{aligned}J\Delta^{1/2}_{\eta\circ\tau_V^t|\omega\circ\tau_V^t}\mathrm{e}^{-\mathrm{i}tL_V}\tau_V^t(A)\Omega_\omega &= \mathrm{e}^{-\mathrm{i}tL_V}\tau_V^t(A)^*\Omega_\eta\\ &= \mathrm{e}^{-\mathrm{i}tL_V}J\Delta^{1/2}_{\eta|\omega}\tau_V^t(A)\Omega_\omega\\ &= J\mathrm{e}^{-\mathrm{i}tL_V}\Delta^{1/2}_{\eta|\omega}\tau_V^t(A)\Omega_\omega,\end{aligned}$$

where we used (2.7). Since $\mathfrak{M}\Omega_\omega$ is a core for $\Delta^{1/2}_{\eta|\omega}$ and $\mathrm{e}^{-\mathrm{i}tL_V}\mathfrak{M}\Omega_\omega = \mathfrak{M}\Omega_{\omega\circ\tau_V^t}$ is a core for $\Delta^{1/2}_{\eta\circ\tau_V^t|\omega\circ\tau_V^t}$, we derive the relation

$$\Delta_{\eta\circ\tau_V^t|\omega\circ\tau_V^t} = \mathrm{e}^{-\mathrm{i}tL_V}\Delta_{\eta|\omega}\mathrm{e}^{\mathrm{i}tL_V}. \tag{2.13}$$

We now deal with $\Delta_{\omega|\omega\circ\tau_V^t}$. First, for any $A\in\mathfrak{M}$,

$$J\Delta^{1/2}_{\omega|\omega\circ\tau_V^t}A\Omega_{\omega\circ\tau_V^t} = A^*\Omega_\omega.$$

Equations (2.8), (2.11) and (2.4) yield that

$$\Omega_{\omega\circ\tau_V^t} = U^*(t)j(U^*(t))\Omega_\omega,$$

and since $j(U^*(t))\in\mathfrak{M}'$, we derive

$$\begin{aligned}J\Delta^{1/2}_{\omega|\omega\circ\tau_V^t}j(U^*(t))AU^*(t)\Omega_\omega &= A^*\Omega_\omega\\ &= U^*(t)(AU^*(t))^*\Omega_\omega\\ &= U^*(t)J\Delta_\omega^{1/2}AU^*(t)\Omega_\omega\\ &= Jj(U^*(t))\Delta_\omega^{1/2}AU^*(t)\Omega_\omega.\end{aligned}$$

Since $\mathfrak{M}\Omega_\omega$ is a core of $\Delta_\omega^{1/2}$ and $j(U^*(t))\mathfrak{M}\Omega_\omega = \mathfrak{M}\Omega_{\omega\circ\tau_V^t}$ is a core of $\Delta^{1/2}_{\omega|\omega\circ\tau_V^t}$ we conclude that

$$\Delta_{\omega|\omega\circ\tau_V^t} = j(U^*(t))\Delta_\omega j(U(t)). \tag{2.14}$$

Going back to (2.12), we derive from Equations (2.13) and (2.14)

$$\begin{aligned}[D\omega : D\eta\circ\tau_V^t]^s &= j(U^*(t))\Delta_\omega^{\mathrm{i}s} j(U(t))\mathrm{e}^{-\mathrm{i}tL_V}\Delta_{\eta|\omega}^{-\mathrm{i}s}\mathrm{e}^{\mathrm{i}tL_V}\\ &= j(U^*(t))\Delta_\omega^{\mathrm{i}s} U^*(t)\mathrm{e}^{-\mathrm{i}tL}\Delta_{\eta|\omega}^{-\mathrm{i}s}\mathrm{e}^{\mathrm{i}tL_V}\\ &= j(U^*(t))\sigma_\omega^s(U^*(t))\mathrm{e}^{-\mathrm{i}tL}\Delta_\omega^{\mathrm{i}s}\Delta_{\eta|\omega}^{-\mathrm{i}s}\mathrm{e}^{\mathrm{i}tL_V},\end{aligned}$$

where we used (2.11) and (2.5). Since $\sigma_\omega^s(U^*(t)) \in \mathfrak{M}$, it commutes with $j(U^*(t))$ and another application of (2.11) (together with Relation (2.12) at $t = 0$) gives

$$[D\omega : D\eta\circ\tau_V^t]^s = \sigma_\omega^s(U^*(t))U(t)\tau_V^{-t}([D\omega : D\eta]^s). \tag{2.15}$$

We can therefore write

$$\eta\circ\tau_V^t([D\omega : D\eta\circ\tau_V^t]^s) = (\mathrm{e}^{\mathrm{i}tL_V}U^*(t)\sigma_\omega^s(U(t))\mathrm{e}^{-\mathrm{i}tL_V}\Omega_\eta, [D\omega : D\eta]^s\Omega_\eta). \tag{2.16}$$

Since $U(t) \in \mathcal{D}(\delta_\omega)$, the estimate

$$\mathrm{e}^{\mathrm{i}tL_V}U^*(t)\sigma_\omega^s(U(t))\mathrm{e}^{-\mathrm{i}tL_V}\Omega_\eta = \Omega_\eta + s\mathrm{e}^{\mathrm{i}tL_V}U^*(t)\delta_\omega(U(t))\mathrm{e}^{-\mathrm{i}tL_V}\Omega_\eta + \mathrm{o}(s),$$

holds in the norm of $\mathcal{H}_\omega$ as $s \downarrow 0$. Furthermore, Equation (2.10) is easily rewritten as

$$\mathrm{e}^{\mathrm{i}tL_V}U^*(t)\delta_\omega(U(t))\mathrm{e}^{-\mathrm{i}tL_V} = \mathrm{i}\int_0^t \pi_\omega(\tau_V^u(\sigma_V))du.$$

Equation (2.16) leads to the estimate

$$\begin{aligned}\eta\circ\tau_V^t([D\omega : D\eta\circ\tau_V^t]^s) &= \eta([D\omega : D\eta]^s)\\ &\quad - \mathrm{i}s\int_0^t (\Omega_\eta, \pi_\omega(\tau_V^u(\sigma_V))[D\omega : D\eta]^s\Omega_\eta)\,du + \mathrm{o}(s),\end{aligned} \tag{2.17}$$

as $s \downarrow 0$. Since the cocyle $[D\omega : D\eta]^s$ is strongly continuous, insertion in Equation (2.9) gives the result. □

Proof of Proposition 1.3. For any self-adjoint $P \in \mathfrak{M}$ we define a group of automorphisms of $\mathfrak{M}$ by

$$\sigma_P^t(A) = \mathrm{e}^{\mathrm{i}t(\mathcal{L}_\omega+P)}A\mathrm{e}^{-\mathrm{i}t(\mathcal{L}_\omega+P)}.$$

Araki's perturbation theory yields that there is a state $\omega_P \in \mathcal{N}_\omega$ which is a $(\sigma_P, -1)$-KMS state. Let $\mathcal{N}'_\omega$ be the set of all states obtained in this manner. It is well-known that $\mathcal{N}'_\omega$ is dense in $\mathcal{N}_\omega$ (see, e.g., [R]). By the result of Araki (see Proposition 6.2.32 in [BR2]),

$$\mathrm{Ent}(\eta\circ\tau_V^t\,|\,\omega_P) = \mathrm{Ent}(\eta\circ\tau_V^t\,|\,\omega) + \eta(\tau_V^t(P)) - \log\|\mathrm{e}^{(\mathcal{L}_\omega+P)/2}\Omega_\omega\|^2.$$

The statement follows from this relation, the obvious estimate $|\eta(\tau_V^t(P))| \leq \|P\|$ and the fact that $0 < \|e^{(\mathcal{L}_\omega + P)/2}\Omega_\omega\| < \infty$. □

Proof of Proposition 1.4. Since $\eta \circ \tau_V^t = \eta$, Relation (2.17) yields that for all $s > 0$,

$$\int_0^t (\Omega_\eta, \pi_\omega(\tau_V^u(\sigma_V))[D\omega : D\eta]^s \Omega_\eta)\, du = o(1).$$

Taking $s \downarrow 0$ we get that for all t,

$$\int_0^t \eta(\sigma_V) du = t\eta(\sigma_V) = 0,$$

and so $\mathrm{Ep}_V(\eta) = 0$. □

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