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Spectral Structure of Anderson Type Hamiltonians

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Abstract. We study self adjoint operators of the form $H_\omega = H_0 + \sum \lambda_\omega(n)\langle\delta_n, \cdot\rangle\delta_n$, where the δ_n's are a family of orthonormal vectors and the $\lambda_\omega(n)$'s are independently distributed random variables with absolutely continuous probability distributions. We prove a general structural theorem saying that for each pair (n,m), if the cyclic subspaces corresponding to the vectors δ_n and δ_m are not completely orthogonal, then the restrictions of H_ω to these subspaces are unitarily equivalent (with probability one). This has some consequences for the spectral theory of such operators. In particular, we show that "well behaved" absolutely continuous spectrum of Anderson type Hamiltonians must be pure, and use this to prove the purity of absolutely continuous spectrum in some concrete cases.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and H_0 a bounded self adjoint operator on $\mathcal{H}$. Let $\{\delta_n\}_{n\in\mathcal{N}} \subset \mathcal{H}$ be a set of orthonormal vectors, where $\mathcal{N}$ is either finite or a countable infinite set. Let $\{p_n\}_{n\in\mathcal{N}}$ be absolutely continuous (w.r.t. Lebesgue measure) probability measures on $\mathbb{R}$, and consider the probability space (Ω, dP), where $\Omega = \times_{\mathcal{N}} \mathbb{R}$ and $dP = \times_{\mathcal{N}} dp_n$. For each $\omega = (\lambda_\omega(1), \lambda_\omega(2), \ldots) \in \Omega$, define

$$H_\omega = H_0 + \sum_{n\in\mathcal{N}} \lambda_\omega(n)\langle\delta_n, \cdot\rangle\delta_n \,. \tag{1.1}$$

Operators of the form (1.1) often arise as discrete Schrödinger operators with random potentials in models of condensed matter physics. Perhaps the most famous example of this type is the d-dimensional Anderson model, which has the form (1.1) on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, with $H_0 = \Delta$ being the discrete Laplacian on $\ell^2(\mathbb{Z}^d)$ ($(\Delta\psi)(n) = \sum_{|j-n|=1} \psi(j)$), $\mathcal{N} = \mathbb{Z}^d$, the δ_n's being the standard basis of $\ell^2(\mathbb{Z}^d)$ (namely, $\delta_n(m) = \delta_{nm}$, where δ_{nm} is the Kronecker delta), and $p_n = p$ for some fixed measure p and all n's. There are also many variants of this model, such as operators where H_0 is the Laplacian plus some fixed potential, operators where the potential is supported on a subset of $\mathbb{Z}^d$, operators where the potential is random-decaying (e.g., $dp_n(\lambda) = f(a_n\lambda)\, d(a_n\lambda)$ for some fixed f and $a_n \to \infty$ as $|n| \to \infty$), operators on graphs other than $\mathbb{Z}^d$, etc... While we discuss below some concrete examples of this kind, our main results here involve general random operators of the form (1.1), and are thus valid in this more general context.

Our main result in this paper is the following structural theorem:

Theorem 1.1. *Let $\{H_\omega\}_{\omega\in\Omega}$ be a family of operators of the form given by (1.1), and for each $n \in \mathcal{N}$ let $\mathcal{H}_{\omega,n}$ be the cyclic subspace spanned by H_ω and δ_n. Let $n, m \in \mathcal{N}$, and suppose that for a.e. $\omega \in \Omega$, the subspaces $\mathcal{H}_{\omega,n}$ and $\mathcal{H}_{\omega,m}$ are not orthogonal. Then for a.e. $\omega \in \Omega$, the restrictions $H_\omega \restriction \mathcal{H}_{\omega,n}$ and $H_\omega \restriction \mathcal{H}_{\omega,m}$ are unitarily equivalent.*

Remarks. 1. One easily verifies that for any $\omega \in \Omega$, H_ω is essentially self adjoint. The cyclic subspaces are defined by $\mathcal{H}_{\omega,n} = \overline{\{f(H_\omega)\delta_n \,|\, f \in C_\infty(\mathbb{R})\}}$, where $\overline{\cdot}$ denotes a closure, and $C_\infty(\mathbb{R})$ is the set of (complex valued) continuous functions on $\mathbb{R}$ with the property that for any $\epsilon > 0$, there exists a compact set $D_\epsilon \subset \mathbb{R}$ such that $|f(x)| < \epsilon$ if $x \notin D_\epsilon$.

2. The assumption that H_0 is bounded can be relaxed, as long as we separately require that the δ_n's obey $\delta_n \in \mathcal{D}(H_0)$ (where $\mathcal{D}(\cdot)$ denotes the domain of the operator), and that the H_ω's are essentially self adjoint (for a.e. $\omega \in \Omega$).

3. The assumption that the $\lambda_\omega(n)$'s are completely independent can also be relaxed. Theorem 1.1 (along with its Corollaries 1.1.1–1.1.3 below) would remain true if we consider any probability measure P on Ω that has the property that for each n, the conditional probability distribution of $\lambda_\omega(n)$, given any $\{\lambda_\omega(m)\}_{m\neq n}$, is absolutely continuous.

4. We note that by the spectral theorem [18], each of the restriction operators $H_\omega \restriction \mathcal{H}_{\omega,n}$ is unitarily equivalent to multiplication by E on $L^2(\mathbb{R}, d\mu_{\omega,\delta_n})$, where μ_{ω,δ_n} is the spectral measure for H_ω and δ_n, namely, the unique (regular) Borel measure on $\mathbb{R}$ obeying $\langle \delta_n, f(H_\omega)\delta_n\rangle = \int f(E)\, d\mu_{\omega,\delta_n}(E)$ for any bounded Borel function f. Thus, the unitary equivalence of $H_\omega \restriction \mathcal{H}_{\omega,n}$ and $H_\omega \restriction \mathcal{H}_{\omega,m}$ is the same as the mutual equivalence of the two spectral measures μ_{ω,δ_n} and μ_{ω,δ_m} (where we say that two measures are equivalent if they have the same sets of zero measure).

While we are mainly interested here in random operators of the form (1.1), and we have thus formulated our main result in this context, we will see that the core of Theorem 1.1 is, in fact, a theorem about rank-two perturbations which is itself an immediate consequence of a theorem about rank-one perturbations. Recall that the (by now, classical) theory of rank-one perturbations [21] deals with families $\{H_\lambda\}_{\lambda\in\mathbb{R}}$ of self adjoint operators of the form $H_\lambda = H_0 + \lambda\langle\psi,\cdot\rangle\psi$. The theory concentrates on the cyclic subspace spanned by H_λ and ψ (which is independent of λ) and deals with the behavior of the spectral measure $\mu_{\lambda,\psi}$ (for H_λ and ψ) as λ is varied. In this paper we treat a somewhat different question in the same context: We will consider an additional vector φ, and show that if the cyclic subspace spanned by H_λ and ψ is not orthogonal to the cyclic subspace spanned by H_λ and φ, then for Lebesgue a.e. λ, the spectral measure $\mu_{\lambda,\psi}$ is absolutely continuous with respect to the spectral measure $\mu_{\lambda,\varphi}$. This fact is the core of Theorem 1.1. Our proof of it is an extension of an argument of Simon [20] (see more on this below).

In what follows, we are mostly interested in cases where the family $\{\delta_n\}_{n\in\mathcal{N}}$ is a cyclic family for the H_ω's. Given a self adjoint operator H on $\mathcal{H}$ and a (finite or infinite) family of orthonormal vectors $\{\varphi_n\}_{n\in\mathcal{I}}$ (where $\mathcal{I}\subset\mathbb{N}$), we denote by $\mathcal{H}_{\varphi_n}$ the cyclic subspace spanned by H and φ_n, and we say that $\{\varphi_n\}_{n\in\mathcal{I}}$ is a cyclic family for H if the set of all finite sums $\{\tilde{\varphi_1} + \tilde{\varphi_2} + \ldots + \tilde{\varphi_N} \,|\, \tilde{\varphi_n} \in \mathcal{H}_{\varphi_n}\}$ is dense in $\mathcal{H}$. We note that such cyclic families always exists, since every orthonormal basis of $\mathcal{H}$ is a cyclic family for H. Given such a cyclic family, we define a Borel measure μ on $\mathbb{R}$ by

$$\mu = \sum_{n \in \mathcal{I}} 2^{-n} \mu_{\varphi_n} \,, \tag{1.2}$$

where for each n, μ_{φ_n} is the spectral measure for H and φ_n. Such a measure μ completely determines the spectral properties of H, since any spectral measure of H must be absolutely continuous with respect to it. In particular, the Borel decomposition $\mu = \mu_{\rm ac} + \mu_{\rm sc} + \mu_{\rm pp}$ of μ into absolutely continuous, singular continuous, and pure point parts determines the corresponding spectra of H. The absolutely continuous spectrum $\sigma_{\rm ac}(H)$, singular continuous spectrum $\sigma_{\rm sc}(H)$, and pure point spectrum $\sigma_{\rm pp}(H)$, are the topological supports of the corresponding parts of μ. We call the class of measures that are equivalent to μ (namely, those measures having the same sets of zero measure) the *spectral measure class* of the operator H.

Theorem 1.1 immediately implies the following:

Corollary 1.1.1 *Let $\{H_\omega\}_{\omega\in\Omega}$ be as in Theorem* 1.1*, and suppose that for a.e. $\omega \in \Omega$, the following two conditions hold:*
(i) *The family $\{\delta_n\}_{n\in\mathcal{N}}$ is a cyclic family for H_ω.*
(ii) *For every $n, m \in \mathcal{N}$, the subspaces $\mathcal{H}_{\omega,n}$ and $\mathcal{H}_{\omega,m}$ are not orthogonal.*
Then for a.e. $\omega \in \Omega$, for every $n \in \mathcal{N}$, the spectral measure μ_{ω,δ_n} (for H_ω and δ_n) is in the spectral measure class of H_ω.

Corollary 1.1.1 indicates that the spectral theory of operators of the form (1.1) (in cases where the assumptions of the Corollary hold) is somewhat simpler than what one might a priory expect, since it is sufficient to study the restriction of H_ω to any of the cyclic subspaces $\mathcal{H}_{\omega,n}$. More importantly, we will show that Corollary 1.1.1 imposes some restrictions on the kind of spectral properties that such operators might have as well as on the behavior of certain spectral objects when ω is varied. It is tempting to think that Corollary 1.1.1 has something to do with the spectrum of H_ω being simple and the δ_n's being cyclic vectors. Indeed, if this where true, then Corollary 1.1.1 would have followed from it. Furthermore, it has been shown by Simon [20] that for $\{H_\omega\}_{\omega\in\Omega}$ as in Corollary 1.1.1, the δ_n's are indeed cyclic vectors (and thus the spectrum is simple) in case that the H_ω's have only pure point spectrum. In fact, our proof of Theorem 1.1 (and thus of Corollary 1.1) is an extension of Simon's argument. In the pure point case, the mutual equivalence of the μ_{ω,δ_n}'s along with the existence of resolution of the identity in terms of normalized eigenvectors imply the cyclicity of the δ_n's. However, this argument breaks down for continuous spectrum. Moreover, we will see below that there are simple examples where the spectrum is not simple, and so Corollary 1.1.1 holds irrespectively of spectral multiplicity issues.

Given an operator H_ω of the form (1.1), let

$$\mu_\omega = \sum_{n \in \mathcal{N}} 2^{-n} \mu_{\omega,\delta_n} \, . \tag{1.3}$$

We will prove the following:

Corollary 1.1.2 *Let $\{H_\omega\}_{\omega\in\Omega}$ be as in Corollary 1.1.1. Then for P-a.e. pair $\omega, \omega' \in \Omega$, the singular parts of the measures μ_ω and $\mu_{\omega'}$ are mutually singular.*

Remarks. 1. Corollary 1.1.2 is related to the known [4,5] fact about Schrödinger operators with ergodic potentials that $\{\omega \in \Omega \,|\, E \text{ is an eigenvalue of } H_\omega\}$ has zero measure (and thus the pure point parts of μ_ω and $\mu_{\omega'}$ are almost surely mutually singular). Of course, our result does not cover general Schrödinger operators with ergodic potentials, while it does cover many operators that are not ergodic. For the case of Schrödinger operators with i.i.d. random potentials (which are ergodic), our result generalizes the above fact by handling the singular parts of the measures rather than just their pure point parts. Deift-Simon [6] have proven precisely the same kind of result (namely, for the singular parts of the measures) for Schrödinger operators with ergodic potentials in one dimension.

2. It is interesting to note that Corollary 1.1.2 can potentially be used to prove the existence (and even purity) of absolutely continuous spectrum, since it would be enough to show the mutual equivalence of spectral measures for different realizations (with positive probability) in order to establish their absolute continuity. Indeed, this strategy, in conjunction with the Deift-Simon analog of Corollary 1.1.2, had been used by Gordon et. al. [7] (and more recently, through direct application of their results, by Jitomirskaya [11]) to establish purely absolutely continuous spectrum for the almost Mathieu operator.

Our next result is a natural complement of Corollary 1.1.2, involving the essential supports of absolutely continuous spectral measures. Given an absolutely continuous Borel measure ν on $\mathbb{R}$, a measurable set A is said to be *an essential support* of ν if it supports ν (namely, if $\nu(\mathbb{R} \setminus A) = 0$) and if any set of strictly smaller Lebesgue measure does not support ν. Equivalently, A is an essential support of ν if and only if there exists $f \in L^1(\mathbb{R}, dx)$ such that $A = \{x \in \mathbb{R} \,|\, f(x) \neq 0\}$ and $d\nu = f(x)\, dx$ (as measures). We note that if A is such an essential support, then every measurable set which differs from

A by a set of zero Lebesgue measure is also an essential support. Thus, such an essential support can be viewed as an equivalence class of measurable sets (where equivalence here means up to sets of zero Lebesgue measure) rather than as some concrete set. However, one can still talk of concrete sets as being (or not being) essential supports. A possible concrete candidate to represent the essential support is the set

$$A = \{E \in \mathbb{R} \mid \lim_{\epsilon \to 0} \epsilon^{-1} \nu(E - \epsilon, E + \epsilon) \text{ exists and is finite and strictly positive}\}. \quad (1.4)$$

We will prove the following:

Corollary 1.1.3 *Let $\{H_\omega\}_{\omega \in \Omega}$ be as in Corollary 1.1.1. Then there exists an ω-independent Borel set $A \subset \mathbb{R}$, such that for a.e. $\omega \in \Omega$, A is an essential support of the absolutely continuous part of μ_ω. Moreover, for any measurable set $\tilde{A}$ that differs from A by a set of zero Lebesgue measure (namely, for any $\tilde{A}$ that is also an essential support of the absolutely continuous part of μ_ω for a.e. $\omega \in \Omega$), we have that for a.e. $\omega \in \Omega$, $\mu_{\omega,\mathrm{sing}}(\tilde{A}) = 0$, where $\mu_{\omega,\mathrm{sing}}$ is the singular part of μ_ω.*

Remarks. 1. The first part of Corollary 1.1.3, namely, the existence of a non-random almost-sure essential support of the absolutely continuous part of μ_ω, is a fairly elementary consequence of Kolmogorov's zero-one law, and its proof does not require Theorem 1.1. An analogous fact is also known [4,5] in the context of Schrödinger operators with ergodic potentials. While this result (for general random operators of the form (1.1)) is fairly elementary and seems to be known to workers in the field, we are not aware of it previously appearing in the literature.

2. Corollary 1.1.3 provides a (weak) sense in which absolutely continuous spectrum of such random operators must be pure, since it says that with probability one, the singular parts of spectral measures must be supported outside the (non-random) essential support of the absolutely continuous spectrum. This does not yet insure truly pure absolutely continuous spectrum, since the spectra themselves are, roughly speaking, closures of the corresponding supports, and so Corollary 1.1.3 still allows for a situation where $\sigma_{\mathrm{ac}}(H_\omega) \cap \sigma_{\mathrm{sing}}(H_\omega) \neq \emptyset$. However, if it so happens that the essential support A of Corollary 1.1.3 contains an open interval I, then it follows that for a.e. $\omega \in \Omega$, H_ω has purely absolutely continuous spectrum on I, namely, $I \subset \sigma_{\mathrm{ac}}(H_\omega)$ and $I \cap \sigma_{\mathrm{sing}}(H_\omega) = \emptyset$.

As we shall see below, Corollary 1.1.3 can be used to establish the purity of absolutely continuous spectrum in many concrete examples, including some cases where this has been an open problem for some time. Furthermore, it might be used in the future to prove the purity of absolutely continuous spectrum in cases where its existence is not currently known (such as the Anderson model). We note at this point that if H_ω is a discrete Laplacian + potential on $\ell^2(\mathcal{G})$, for some connected graph $\mathcal{G}$ (namely, if $H_\omega = \Delta + \sum_{n\in G} V(n)\langle\delta_n,\cdot\rangle\delta_n + \sum_{n\in\mathcal{N}} \lambda_\omega(n)\langle\delta_n,\cdot\rangle\delta_n$, where the δ_n's are delta functions on the graph, and $(\Delta\psi)(n) = \sum_{|n-j|=1}\psi(j)$, where $|n-j|$ denotes the distance on the graph between n and j), then it is always true that for any pair $n,m\in\mathcal{N}$ and $\omega\in\Omega$, the cyclic subspaces $\mathcal{H}_{\omega,n}$ and $\mathcal{H}_{\omega,m}$ are not orthogonal. This can be seen by noting that $\langle H_\omega^{|n-m|}\delta_n,\delta_m\rangle$ always takes a strictly positive integer value that is independent of the potential (more explicitly, it is equal to the number of different paths of length $|n-m|$ that connect n and m on the graph). Thus, if the set of points $\mathcal{N}$ where the random part of the potential lives is sufficiently large so that $\{\delta_n\}_{n\in\mathcal{N}}$ is a cyclic family for the H_ω's, then Corollaries 1.1.1–1.1.3 are fully applicable to such operators.

One of the simplest examples of Schrödinger operators to which Theorem 1.1 can be applied is that of a one-dimensional operator along with two consecutive rank-one perturbations. That is, consider the operator

$$H_{\lambda,\eta} = H_0 + \lambda\langle\delta_0,\cdot\rangle\delta_0 + \eta\langle\delta_1,\cdot\rangle\delta_1 \tag{1.5}$$

on $\ell^2(\mathbb{Z})$, where $H_0 = \Delta + V$ for some fixed potential V (such that $(H_{\lambda,\eta}\psi)(n) = \psi(n+1)+\psi(n-1)+(V(n)+\lambda\delta_{0n}+\eta\delta_{1n})\psi(n)$). Theorem 1.1 implies that for Lebesgue a.e. pair λ,η, the restrictions of $H_{\lambda,\eta}$ to the cyclic subspaces it spans with δ_0 and with δ_1 are unitarily equivalent. It is elementary that if $V=0$, then the above $H_{\lambda,\eta}$ has absolutely continuous spectrum of multiplicity 2 on $[-2,2]$, and so we see that the conclusion of Theorem 1.1 holds regardless of simple spectral multiplicity.

Our next example involves discrete Schrödinger operators with random-decaying potentials. By combining Corollary 1.1.3 with a result of Krishna [13], we have the following:

Theorem 1.2. *Let $d\geq 3$ and let p be an absolutely continuous probability measure on $\mathbb{R}$ with $\int\lambda\, dp(\lambda)=0$ and $\int\lambda^2\, dp(\lambda)<\infty$. Let $\{a_n\}_{n\in\mathbb{Z}^d}\subset\mathbb{R}$ be such that $0<|a_n|<|n|^{-\alpha}$ for some $\alpha>1$. Let $H_\omega=\Delta+V_\omega$ on $\ell^2(\mathbb{Z}^d)$, where the potential V_ω is given by $V_\omega(n)=a_n\lambda_\omega(n)$ and the $\lambda_\omega(n)$'s are independently identically distributed random variables with common probability distribution p. Then for a.e. ω, H_ω has purely absolutely continuous spectrum on $(-2d,2d)$.*

Remarks. 1. Krishna [13] had shown that for H_ω's as in Theorem 1.2, $[-2d,2d]\subset\sigma_{\rm ac}(H_\omega)$ for a.e. ω. His proof is based on showing the existence of wave operators (w.r.t. the free

Laplacian Δ), and it thus also imply that $[-2d, 2d]$ is contained in an essential support of the absolutely continuous part of $d\mu_\omega$. (This is because the existence of these wave operators imply unitary equivalence between the restriction of H_ω to a subspace and the free Laplacian Δ [19].) Thus, Theorem 1.2 (namely, the purity of the absolutely continuous spectrum on $(-2d, 2d)$), follows immediately from Corollary 1.1.3.

2. Kirsch, Krishna, and Obermeit [12] have recently studied operators H_ω as in Theorem 1.2. They show that with appropriate restrictions on the measure p (they need it to have some smoothness properties and not to decay too fast at infinity) and on the a_n's (they should not decay too fast) the resulting H_ω's have (for a.e. ω) spectrum on the entire real line, and moreover, the spectrum outside $[-2d, 2d]$ is purely pure point (namely, $\mathbb{R} \setminus [-2d, 2d] \subset \sigma_{\rm pp}(H_\omega) \setminus \sigma_{\rm c}(H_\omega)$). Their proof relies on earlier works by Aizenman [1] and Aizenman-Molchanov [2]. Our Theorem 1.2 completes the spectral picture for these models by proving the absence of singular spectrum on $(-2d, 2d)$. That is, we get $\sigma_{\rm ac}(H_\omega) = [-2d, 2d]$, $\sigma_{\rm pp}(H_\omega) = \mathbb{R} \setminus (-2d, 2d)$, and $\sigma_{\rm sc}(H_\omega) = \emptyset$.

Another example we wish to discuss here is that of Laplacians on a "half space" with a random boundary potential. In [10] we prove the following:

Theorem 1.3. *Let $d \geq 2$ and let $\{H_\omega\}_{\omega\in\Omega}$ be Schrödinger operators of the form $\Delta + V_\omega$ on $\ell^2(\mathbb{Z}^{d-1} \times \mathbb{Z}_+)$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and V_ω is a random potential supported on the boundary. That is*

$$V_\omega(n, m) = \begin{cases} \lambda_\omega(n) & m = 0 \\ 0 & m \neq 0\,, \end{cases}$$

where $n \in \mathbb{Z}^{d-1}$, $m \in \mathbb{Z}_+$, and the $\lambda_\omega(n)$'s are independently distributed (real) random variables with absolutely continuous probability distributions. Then, for a.e. $\omega \in \Omega$, H_ω has purely absolutely continuous spectrum on $(-2d, 2d)$.

Remarks. 1. Our proof of Theorem 1.3 (given in [10]) is based on proving the existence of wave operators w.r.t. the free Laplacian Δ on $\ell^2(\mathbb{Z}^{d-1} \times \mathbb{Z}_+)$. It thus yields that for a.e. ω, $[-2d, 2d]$ is in an essential support of the absolutely continuous part of μ_ω. Since the set of delta function vectors on the boundary is easily seen to form a cyclic family for the H_ω's, Theorem 1.3 follows as an immediate consequence of Corollary 1.1.3.

2. It is possible to construct a compact deterministic potential V that is supported on the boundary of $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$, so that $\Delta + V$ on $\ell^2(\mathbb{Z}^{d-1} \times \mathbb{Z}_+)$ will have an eigenvalue (imbedded in the absolutely continuous spectrum) in $(-2d, 2d)$ [15]. Thus, the purity of the

absolutely continuous spectrum on $(-2d, 2d)$ in the random case can't follow from a simple perturbation bound. Theorem 1.3 says that the occurrence of such imbedded eigenvalues is sufficiently rare so that they occur with probability 0 in appropriate random settings.

3. Jakšić-Molchanov [8,9] have recently studied operators as in Theorem 1.3. They show that there are cases where such operators have Anderson localization (namely, purely pure point essential spectrum) outside $[-2d, 2d]$. Moreover, there are cases where all spectrum outside $[-2d, 2d]$ is purely pure point, and so, using our Theorem 1.3, one gets a complete spectral picture with purely absolutely continuous spectrum on $(-2d, 2d)$, a pure point component (in some cases $\mathbb{R} \setminus (-2d, 2d)$) outside $(-2d, 2d)$, and no singular continuous spectrum.

The rest of this paper is organized as follows. In Section 2 we prove Theorem 1.1, in Section 3 we prove Corollary 1.1.2, and in Section 4 we prove Corollary 1.1.3.

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2. Proof of Theorem 1.1

We start with some background facts.

Proposition 2.1. *Let H be a self adjoint operator on $\mathcal{H}$ and let $\varphi, \psi \in \mathcal{H}$. Suppose that the cyclic subspaces $\mathcal{H}_\varphi$ and $\mathcal{H}_\psi$, spanned by H and, correspondingly, φ and ψ, are not orthogonal. Then for Lebesgue a.e. $E \in \mathbb{R}$, the limit*

$$\lim_{\epsilon \to 0} \langle \varphi, (H - E - i\epsilon)^{-1} \psi \rangle \equiv \langle \varphi, (H - E - i0)^{-1} \psi \rangle$$

exists and is finite and non-zero.

Remark. Note that Proposition 2.1 does not exclude the case $\varphi = \psi$.

Proof. By the spectral theorem [18], we have (for any $z \in \mathbb{C}$)

$$\langle \varphi, (H - z)^{-1} \psi \rangle = \int \frac{d\mu_{\varphi,\psi}(x)}{x - z}, \tag{2.1}$$

where $\mu_{\varphi,\psi}$ is an appropriate (finite, complex valued) spectral measure. That is, $\langle\varphi,(H-z)^{-1}\psi\rangle$ is the Borel (a.k.a. Stieltjes) transform of a finite complex valued measure and, in particular, it is an analytic function of z in the upper half plane. By mapping the upper half plane to the unit disc and using known theorems about boundary values of analytic functions in the disk [14] (also see Appendix A of [16] and references therein), one deduces that $\langle\varphi,(H-E-i0)^{-1}\psi\rangle$ exists and is finite for a.e. $E\in\mathbb{R}$. Moreover, it must also be non-zero for a.e. $E\in\mathbb{R}$, unless $\langle\varphi,(H-z)^{-1}\psi\rangle$ vanishes identically (as a function of $z\in\mathbb{C}$), and this happens if and only if the measure $\mu_{\varphi,\psi}$ vanishes identically. But, if $\mu_{\varphi,\psi}$ vanishes, then we have

$$\langle g(H)\varphi, f(H)\psi\rangle = \int g^*(x)f(x)\,d\mu_{\varphi,\psi}(x) = 0\,, \tag{2.2}$$

for any $f,g\in C_\infty(\mathbb{R})$, and so the cyclic subspaces $\mathcal{H}_\varphi$ and $\mathcal{H}_\psi$ are orthogonal. $\square$

Next, we need to recall here a few basic facts from the classical theory of rank one perturbations (see [21] for a proof):

Proposition 2.2. *Let H_0 be a self adjoint operator on $\mathcal{H}$ and let $\psi\in\mathcal{H}$. For each $\lambda\in\mathbb{R}$, let*

$$H_\lambda = H_0 + \lambda\langle\psi,\cdot\rangle\psi\,,$$

and let $\mu_{\lambda,\psi}$ be the spectral measure for H_λ and ψ. Then

(i) *The set* $\{E\in\mathbb{R}\,|\,\langle\psi,(H_\lambda-E-i0)^{-1}\psi\rangle$ exists and $0<\operatorname{Im}\langle\psi,(H_\lambda-E-i0)^{-1}\psi\rangle<\infty\}$ *is independent of λ, and for any $\lambda\in\mathbb{R}$, it is an essential support of the absolutely continuous part of $\mu_{\lambda,\psi}$.*

(ii) *The singular part of $\mu_{\lambda,\psi}$ is supported on the set $\{E\in\mathbb{R}\,|\,\langle\psi,(H_0-E-i0)^{-1}\psi\rangle = -\lambda^{-1}\}$.*

(iii) *For any $B\subset\mathbb{R}$ of zero Lebesgue measure, we have $\mu_{\lambda,\psi}(B)=0$ for Lebesgue a.e. $\lambda\in\mathbb{R}$.*

The final background result that we need here is the following theorem of Poltoratskii [17].

Proposition 2.3. *Let μ and ν be two finite complex valued measures on $\mathbb{R}$, and for any $z\in\mathbb{C}$, let $F(\mu,z)\equiv\int(x-z)^{-1}\,d\mu$ and $F(\nu,z)\equiv\int(x-z)^{-1}\,d\nu$ be the corresponding*

Borel transforms. Let $\nu = \nu_1 + \nu_2$ be the Lebesgue decomposition of ν w.r.t. μ, such that ν_1 is absolutely continuous w.r.t. μ and ν_2 is singular w.r.t. μ. Furthermore, let $\nu_{2,\mathrm{sing}}$ be the singular (w.r.t. Lebesgue measure) part of ν_2. Then, for a.e. E w.r.t. $\nu_{2,\mathrm{sing}}$, we have

$$\lim_{\epsilon\to 0} \frac{|F(\mu, E+i\epsilon)|}{|F(\nu, E+i\epsilon)|} = 0\,.$$

Remarks. 1. Poltoratskii discusses measures on the unit circle. The implication for measures on $\mathbb{R}$ is both standard and straight-forward.

2. The result is not explicitly stated in [17] in this way. However, Corollary 1 in page 403 of [17] is exactly our Proposition 2.3 for the case where $\nu = \nu_2 = \nu_{2,\mathrm{sing}}$. The more general form follows immediately from this assertion, because we have for a.e. E w.r.t. $\nu_{2,\mathrm{sing}}$,

$$\lim_{\epsilon\to 0} \frac{|F(\mu, E+i\epsilon)|}{|F(\nu_{2,\mathrm{sing}}, E+i\epsilon)|} = 0$$

and

$$\lim_{\epsilon\to 0} \frac{|F(\nu, E+i\epsilon)|}{|F(\nu_{2,\mathrm{sing}}, E+i\epsilon)|} = \lim_{\epsilon\to 0} \frac{|F(\nu-\nu_{2,\mathrm{sing}}, E+i\epsilon) + F(\nu_{2,\mathrm{sing}}, E+i\epsilon)|}{|F(\nu_{2,\mathrm{sing}}, E+i\epsilon)|} = 1\,.$$

Our next theorem is a general result concerning rank one perturbations of self adjoint operators. In essence, it is the main result of this paper.

Theorem 2.4. *Let H_0 be a self adjoint operator on $\mathcal{H}$ and let $\varphi, \psi \in \mathcal{H}$. For each $\lambda \in \mathbb{R}$, let*

$$H_\lambda = H_0 + \lambda\langle\psi, \cdot\rangle\psi\,,$$

and let $\mu_{\lambda,\varphi}$ and $\mu_{\lambda,\psi}$ be the spectral measures for H_λ and, correspondingly, φ and ψ. Suppose that the cyclic subspaces $\mathcal{H}_{\lambda,\varphi}$ and $\mathcal{H}_{\lambda,\psi}$, spanned by H_λ and, correspondingly, φ and ψ, are not orthogonal. Then for Lebesgue a.e. $\lambda \in \mathbb{R}$, $\mu_{\lambda,\psi}$ is absolutely continuous w.r.t. $\mu_{\lambda,\varphi}$, namely, there exists $f_{\lambda,\psi,\varphi} \in L^1(\mathbb{R}, d\mu_{\lambda,\varphi})$ such that $d\mu_{\lambda,\psi} = f_{\lambda,\psi,\varphi} d\mu_{\lambda,\varphi}$.

Remark. Note that $\mathcal{H}_{\lambda,\psi}$ is independent of λ, and while $\mathcal{H}_{\lambda,\varphi}$ does, in general, depend on λ, it will be independent of λ in case that it is orthogonal to $\mathcal{H}_{\lambda,\psi}$ for some $\lambda \in \mathbb{R}$. Thus, the non-orthogonality of $\mathcal{H}_{\lambda,\varphi}$ and $\mathcal{H}_{\lambda,\psi}$ is a λ-independent fact.

Proof. ¿From the general operator formula $A^{-1} - B^{-1} = B^{-1}(B-A)A^{-1}$, we get for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$(H_\lambda - z)^{-1} = (H_0 - z)^{-1} - \lambda (H_0 - z)^{-1}(\langle \psi, \cdot \rangle \psi)(H_\lambda - z)^{-1}, \tag{2.3}$$

which implies that for any $\phi_1, \phi_2 \in \mathcal{H}$ we have

$$\langle \phi_1, (H_\lambda - z)^{-1}\phi_2 \rangle = \langle \phi_1, (H_0 - z)^{-1}\phi_2 \rangle - \lambda \langle \phi_1, (H_0 - z)^{-1}\psi \rangle \langle \psi, (H_\lambda - z)^{-1}\phi_2 \rangle. \tag{2.4}$$

In what follows, it would be convenient to have more compact notations for the various resolvent matrix elements. We thus set, for any $\lambda \in \mathbb{R}$, $\phi_1, \phi_2 \in \mathcal{H}$, and $z \in \mathbb{C}$, $G_\lambda(\phi_1, \phi_2, z) \equiv \langle \phi_1, (H_\lambda - z)^{-1}\phi_2 \rangle$. Thus, (2.4) can be rewritten as

$$G_\lambda(\phi_1, \phi_2, z) = G_0(\phi_1, \phi_2, z) - \lambda G_0(\phi_1, \psi, z) G_\lambda(\psi, \phi_2, z). \tag{2.5}$$

Setting $\phi_1 = \phi_2 = \psi$ in (2.5), we get

$$G_\lambda(\psi, \psi, z) = G_0(\psi, \psi, z) - \lambda G_0(\psi, \psi, z) G_\lambda(\psi, \psi, z), \tag{2.6}$$

which can be rewritten as

$$G_\lambda(\psi, \psi, z) = \frac{G_0(\psi, \psi, z)}{1 + \lambda G_0(\psi, \psi, z)}. \tag{2.7}$$

Since $G_\lambda(\psi, \psi, z)$ is the Borel transform of $\mu_{\lambda,\psi}$, namely,

$$G_\lambda(\psi, \psi, z) = \int (x - z)^{-1}\, d\mu_{\lambda,\psi}(x), \tag{2.8}$$

(2.7) relates the perturbed spectral measure $\mu_{\lambda,\psi}$ to $\mu_{0,\psi}$. It is the fundamental formula for developing the theory of rank one perturbations [21] (and in particular, for proving Proposition 2.2).

We will now use (2.5) to develop relations between $\mu_{\lambda,\psi}$ and $\mu_{\lambda,\varphi}$. By setting $\phi_1 = \psi$ and $\phi_2 = \varphi$ in (2.5), we obtain (similarly to (2.7))

$$G_\lambda(\psi, \varphi, z) = \frac{G_0(\psi, \varphi, z)}{1 + \lambda G_0(\psi, \psi, z)}, \tag{2.9}$$

and by setting $\phi_1 = \phi_2 = \varphi$ in (2.5), we get

$$G_\lambda(\varphi, \varphi, z) = G_0(\varphi, \varphi, z) - \lambda G_0(\varphi, \psi, z) G_\lambda(\psi, \varphi, z). \tag{2.10}$$

Inserting $G_\lambda(\psi,\varphi,z)$ from (2.9) in (2.10), we get

$$G_\lambda(\varphi,\varphi,z) = G_0(\varphi,\varphi,z) - \lambda\frac{G_0(\varphi,\psi,z)G_0(\psi,\varphi,z)}{1+\lambda G_0(\psi,\psi,z)}\,. \tag{2.11}$$

By the Lebesgue decomposition theorem, we always have a decomposition of the form,

$$d\mu_{\lambda,\psi} = f_{\lambda,\psi,\varphi}d\mu_{\lambda,\varphi} + d\tilde{\mu}_{\lambda,\psi,\varphi}\,, \tag{2.12}$$

where $\tilde{\mu}_{\lambda,\psi,\varphi}$ is the part of $\mu_{\lambda,\psi}$ which is singular w.r.t. $\mu_{\lambda,\varphi}$. The Theorem would thus follow if we can show that for Lebesgue a.e. $\lambda\in\mathbb{R}$, $\tilde{\mu}_{\lambda,\psi,\varphi}=0$.

Let A_1 be the set of all $E\in\mathbb{R}$ for which the limits $G_0(\psi,\psi,E+i0)$, $G_0(\varphi,\psi,E+i0)$, $G_0(\psi,\varphi,E+i0)$, and $G_0(\varphi,\varphi,E+i0)$ exist and are finite and non-zero. By Proposition 2.1, A_1 is a set of full Lebesgue measure, and thus by (iii) of Proposition 2.2, we have $\mu_{\lambda,\psi}=\mu_{\lambda,\psi}\restriction A_1$ for Lebesgue a.e. $\lambda\in\mathbb{R}$. Thus, it suffices to analyze the restriction of the various measures to the set A_1. In order to show that $\tilde{\mu}_{\lambda,\psi,\varphi}$ vanishes (for a.e. $\lambda\in\mathbb{R}$), we will use two separate treatments. One for the singular part of $\mu_{\lambda,\psi}$ and the other for its absolutely continuous part. We start with the singular part.

By rearranging (2.7), we have

$$\frac{1}{1+\lambda G_0(\psi,\psi,z)} = \frac{G_\lambda(\psi,\psi,z)}{G_0(\psi,\psi,z)}\,, \tag{2.13}$$

and by inserting $1+\lambda G_0(\psi,\psi,z)$ from (2.13) in (2.11), we get

$$G_\lambda(\varphi,\varphi,z) = G_0(\varphi,\varphi,z) - \lambda\frac{G_0(\varphi,\psi,z)G_0(\psi,\varphi,z)}{G_0(\psi,\psi,z)}G_\lambda(\psi,\psi,z)\,. \tag{2.14}$$

(2.14) implies that for any $\lambda\in\mathbb{R}$ and $E\in A_1$,

$$\lim_{\epsilon\to 0}\frac{G_\lambda(\varphi,\varphi,E+i\epsilon)}{G_\lambda(\psi,\psi,E+i\epsilon)} = \lim_{\epsilon\to 0}\frac{G_0(\varphi,\varphi,E+i0)}{G_\lambda(\psi,\psi,E+i\epsilon)} - \lambda\frac{G_0(\varphi,\psi,E+i0)G_0(\psi,\varphi,E+i0)}{G_0(\psi,\psi,E+i0)}\,, \tag{2.15}$$

and since $|G_\lambda(\psi,\psi,E+i\epsilon)|\to\infty$ as $\epsilon\to 0$, for a.e. E w.r.t. the singular part of $\mu_{\lambda,\psi}$, we obtain that

$$\lim_{\epsilon\to 0}\frac{G_\lambda(\varphi,\varphi,E+i\epsilon)}{G_\lambda(\psi,\psi,E+i\epsilon)} = -\lambda\frac{G_0(\varphi,\psi,E+i0)G_0(\psi,\varphi,E+i0)}{G_0(\psi,\psi,E+i0)} \neq 0\,, \tag{2.16}$$

for every $\lambda \neq 0$ and a.e. E w.r.t. the singular part of $\mu_{\lambda,\psi} \restriction A_1$. By Proposition 2.3, this implies that the singular part of $\tilde{\mu}_{\lambda,\psi,\varphi} \restriction A_1$ vanishes for every $\lambda \neq 0$, and thus that the singular part of $\tilde{\mu}_{\lambda,\psi,\varphi}$ vanishes for Lebesgue a.e. $\lambda \in \mathbb{R}$.

It remains to show that the absolutely continuous part of $\tilde{\mu}_{\lambda,\psi,\varphi}$ vanishes. By multiplying both sides of (2.11) by $|1+\lambda G_0(\psi,\psi,z)|^2$ and taking imaginary parts, we obtain

$$\begin{aligned} |1+\lambda G_0(\psi,\psi,z)|^2 \mathrm{Im}\, G_\lambda(\varphi,\varphi,z) = {} & |1+\lambda G_0(\psi,\psi,z)|^2 \mathrm{Im}\, G_0(\varphi,\varphi,z) \\ & - \lambda \mathrm{Im}\left[G_0(\varphi,\psi,z)G_0(\psi,\varphi,z)\right] \\ & + \lambda^2 \Big[\mathrm{Im}\, G_0(\psi,\psi,z)\mathrm{Re}\left[G_0(\varphi,\psi,z)G_0(\psi,\varphi,z)\right] \\ & \quad - \mathrm{Re}\, G_0(\psi,\psi,z)\mathrm{Im}\left[G_0(\varphi,\psi,z)G_0(\psi,\varphi,z)\right]\Big]. \end{aligned} \tag{2.17}$$

For $z \in \mathbb{C} \setminus \mathbb{R}$, the r.h.s. of (2.17) is a second order polynomial in λ, which we denote by $P(z,\lambda)$. For $z = E + i\epsilon$ and $E \in A_1$, it also converges as $\epsilon \to 0$ (uniformly on compact sets) to a limiting (second order in λ) polynomial $P(E+i0,\lambda)$. Let

$$A = \{E \in \mathbb{R} \mid G_0(\psi,\psi,E+i0) \text{ exists and } 0 < \mathrm{Im}\, G_0(\psi,\psi,E+i0) < \infty\}. \tag{2.18}$$

We claim that for $E \in A \cap A_1$, $P(E+i0,\lambda)$ can't vanish identically (as a polynomial in λ). Indeed, since $E \in A$, we must have $|1+\lambda G_0(\psi,\psi,E+i0)|^2 > 0$ for all $\lambda \in \mathbb{R}$. Suppose that both the constant term and the linear term of $P(E+i0,\lambda)$ vanish identically, then we must have $\mathrm{Im}\, G_0(\varphi,\varphi,E+i0) = 0$ and also $\mathrm{Im}\left[G_0(\varphi,\psi,z)G_0(\psi,\varphi,z)\right] = 0$. Thus, $P(E+i0,\lambda)$ reduces to $\lambda^2 \mathrm{Im}\, G_0(\psi,\psi,E+i0)\mathrm{Re}\left[G_0(\varphi,\psi,z)G_0(\psi,\varphi,z)\right]$, and since $G_0(\varphi,\psi,z)G_0(\psi,\varphi,z) \neq 0$ for $E \in A_1$, we see that it doesn't vanish.

Since $P(E+i0,\lambda)$ does not vanish identically for $E \in A \cap A_1$, it follows that for each $E \in A \cap A_1$, it can vanish for at most two values of λ (actually, for at most one value, since the l.h.s. of (2.17) is clearly non-negative). In particular, it follows that for each $E \in A \cap A_1$, $P(E+i0,\lambda) \neq 0$ for Lebesgue a.e. $\lambda \in \mathbb{R}$, and thus by Fubini's theorem, we have that for Lebesgue a.e. $\lambda \in \mathbb{R}$, $P(E+i0,\lambda) \neq 0$ for Lebesgue a.e. $E \in A \cap A_1$. Since $|1+\lambda G_0(\psi,\psi,E+i0)|^2$ exists and is strictly positive for any $\lambda \in \mathbb{R}$ and $E \in A \cap A_1$, it follows that for a.e. $\lambda \in \mathbb{R}$, for a.e. $E \in A \cap A_1$, $\mathrm{Im}\, G_\lambda(\varphi,\varphi,E+i0)$ exists and is finite and strictly positive. Since the absolutely continuous parts of the measures $\mu_{\lambda,\psi}$ and $\mu_{\lambda,\varphi}$ are given by $d\mu_{\lambda,\psi,\mathrm{ac}}(E) = \pi^{-1}\mathrm{Im}\, G_\lambda(\psi,\psi,E+i0)\, dE$ and $d\mu_{\lambda,\varphi,\mathrm{ac}}(E) = \pi^{-1}\mathrm{Im}\, G_\lambda(\varphi,\varphi,E+i0)\, dE$, and since $A \cap A_1$ is an essential support of $\mu_{\lambda,\psi,\mathrm{ac}}$, it follows that for a.e. $\lambda \in \mathbb{R}$, $\mu_{\lambda,\psi,\mathrm{ac}}$ is absolutely continuous w.r.t. $\mu_{\lambda,\varphi,\mathrm{ac}}$. This says that the absolutely continuous part of $\tilde{\mu}_{\lambda,\psi,\varphi}$ must vanish, and it thus completes the proof of Theorem 2.4. □

Remark. By Poltoratskii's Theorem 2.7 of [17], the function $f_{\lambda,\psi,\varphi}$ is determined a.e. w.r.t. the singular part of $\mu_{\lambda,\psi}$ by the limiting value of the ratio of the corresponding Borel transforms. Thus, (2.16) implies that

$$f_{\lambda,\psi,\varphi}(E) = -\frac{G_0(\psi,\psi,E+i0)}{\lambda G_0(\varphi,\psi,E+i0)G_0(\psi,\varphi,E+i0)}\,, \tag{2.19}$$

for a.e. $\lambda \in \mathbb{R}$, for a.e. E w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$. Moreover, similarly to (2.15)–(2.16), (2.14) and Proposition 2.3 can be used to show that $\mu_{\lambda,\varphi,\mathrm{sing}} \restriction A_1$ is absolutely continuous w.r.t. $\mu_{\lambda,\psi,\mathrm{sing}}$. Thus, by (ii) of Proposition 2.2 (and noting that $G_0(\varphi,\psi,E+i0)G_0(\psi,\varphi,E+i0) = |G_0(\psi,\varphi,E+i0)|^2$ if $G_0(\psi,\psi,E+i0) \in \mathbb{R}$), this implies that we have

$$d\mu_{\lambda,\psi,\mathrm{sing}}(E) = \frac{\chi_{A_1}(E)}{\lambda^2|G_0(\psi,\varphi,E+i0)|^2}d\mu_{\lambda,\varphi,\mathrm{sing}}(E)\,, \tag{2.20}$$

for Lebesgue a.e. $\lambda \in \mathbb{R}$. We note that a simple relation of the type (2.20) does not hold for the absolutely continuous parts of the measures. The relation for these parts is more complex.

Our next theorem is an immediate consequence of Theorem 2.4 for the case of two independent rank one perturbations.

Theorem 2.5. *Let H_0 be a self adjoint operator on $\mathcal{H}$ and let $\psi,\varphi \in \mathcal{H}$. For every $\lambda,\eta \in \mathbb{R}$, let*

$$H_{\lambda,\eta} = H_0 + \lambda\langle\psi,\cdot\rangle\psi + \eta\langle\varphi,\cdot\rangle\varphi\,,$$

and let $\mu_{\lambda,\eta,\psi}$ and $\mu_{\lambda,\eta,\varphi}$ be the spectral measures for $H_{\lambda,\eta}$ and, correspondingly, ψ and φ. Suppose that the cyclic subspaces $\mathcal{H}_{\lambda,\eta,\psi}$ and $\mathcal{H}_{\lambda,\eta,\varphi}$, spanned by $H_{\lambda,\eta}$ and, correspondingly, ψ and φ, are not orthogonal. Then for Lebesgue a.e. $\lambda,\eta \in \mathbb{R}$, $\mu_{\lambda,\eta,\psi}$ and $\mu_{\lambda,\eta,\varphi}$ are equivalent.

Remark. Note that (similarly to the remark to Theorem 2.4) the non-orthogonality of $\mathcal{H}_{\lambda,\eta,\psi}$ and $\mathcal{H}_{\lambda,\eta,\varphi}$ is independent of λ and η (namely, it holds for any $\lambda,\eta \in \mathbb{R}$ if and only if it holds for $\lambda = \eta = 0$).

Proof. By Theorem 2.4, we have for each fixed η, that $\mu_{\lambda,\eta,\psi}$ is absolutely continuous w.r.t. $\mu_{\lambda,\eta,\varphi}$ for Lebesgue a.e. λ. Thus, by Fubini's theorem, $\mu_{\lambda,\eta,\psi}$ is absolutely continuous w.r.t. $\mu_{\lambda,\eta,\varphi}$ for Lebesgue a.e. pair λ, η. By the same argument we also have that $\mu_{\lambda,\eta,\varphi}$ is absolutely continuous w.r.t. $\mu_{\lambda,\eta,\psi}$ for Lebesgue a.e. pair λ, η. Thus, we have for Lebesgue a.e. $\lambda, \eta \in \mathbb{R}$, that each of the measures $\mu_{\lambda,\eta,\psi}$ and $\mu_{\lambda,\eta,\varphi}$ is absolutely continuous w.r.t. the other, namely, they are equivalent. □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. For every fixed pair $n, m \in \mathcal{N}$, the conditional probability distribution of the pair $(\lambda_\omega(n), \lambda_\omega(m))$, given any $\{\lambda_\omega(k)\}_{k \neq n,m}$, is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}^2$. Thus, it follows from Theorem 2.5 that the corresponding spectral measures, μ_{ω,δ_n} and μ_{ω,δ_m} must be equivalent for a.e. $\omega \in \Omega$. As discussed in Remark 4 to Theorem 1.1, this equivalence of the spectral measures is equivalent to the unitary equivalence of $H_\omega \restriction \mathcal{H}_{\omega,n}$ and $H_\omega \restriction \mathcal{H}_{\omega,m}$. □

3. Proof of Corollary 1.1.2

Proof of Corollary 1.1.2. Suppose that the assertion of the Corollary is not true. Then there must be a Borel set $S \subset \mathbb{R}$ of zero Lebesgue measure and a measurable subset $\tilde{\Omega} \subset \Omega$ with $P(\tilde{\Omega}) > 0$, such that $\mu_\omega(S) > 0$ for every $\omega \in \tilde{\Omega}$. By Corollary 1.1.1, we must thus also have $\mu_{\omega,\delta_1}(S) > 0$ for a.e. $\omega \in \tilde{\Omega}$. Since the conditional probability distribution of $\lambda_\omega(1)$, given any $\{\lambda_\omega(m)\}_{m>1}$, is absolutely continuous w.r.t. Lebesgue measure, there is a subset $\hat{\Omega} \subset \tilde{\Omega}$ for which $\{\lambda_\omega(m)\}_{m>1}$ are fixed, $\lambda_\omega(1)$ varies over a set of positive Lebesgue measure, and $\mu_{\omega,\delta_1}(S) > 0$ for every $\omega \in \hat{\Omega}$. This is a contradiction to (iii) of Proposition 2.2. □

4. Proof of Corollary 1.1.3

Proof of Corollary 1.1.3. For each $\omega \in \Omega$, let A_ω be an essential support of the absolutely continuous part of μ_ω (A_ω can be explicitly chosen by (1.4)). Define a function $f_\omega \in \text{Ł}^1(\mathbb{R}, dE)$ by $f_\omega \equiv \chi_{A_\omega}(E)(1+E^2)^{-1}$. By (for example) Lemma V.2.10 of [4], the map $\Omega \ni \omega \mapsto f_\omega \in \text{Ł}^1(\mathbb{R}, dE)$ is measurable. Since the essential support of the absolutely continuous part of μ_ω is invariant under rank one (and thus finite rank) perturbations, f_ω (as an element of $L^1(\mathbb{R}, dE)$) is independent of $\{\lambda_\omega(n)\}_{n<N}$ for any $N \in \mathcal{N}$. We claim that by Kolmogorov's 0-1 law (see, e.g., [3]), this implies that f_ω (as an element of $L^1(\mathbb{R}, dE)$) is P-almost surely independent of ω. To see this precisely, define

$$F_\omega(x) = \int_{-\infty}^{x} f_\omega(E)\, dE\,, \tag{4.1}$$

and for every $q, r \in \mathbb{Q}$, let $\tilde{\Omega}_q(r) \equiv \{\omega \in \Omega \,|\, F_\omega(q) < r\}$. Since $F_\omega(q)$ is independent of $\{\lambda_\omega(n)\}_{n<N}$, we have, by Kolmogorov's 0-1 law, that for every q, r, either $P(\tilde{\Omega}_q(r)) = 1$ or $P(\tilde{\Omega}_q(r)) = 0$. For every $q \in \mathbb{Q}$, let $\alpha(q) = \inf\{r \in \mathbb{Q} \,|\, P(\tilde{\Omega}_q(r)) = 1\}$. Then $F_\omega(q) = \alpha(q)$ P-almost surely. Since $F_\omega(\cdot)$ is a continuous function on $\mathbb{R}$, there must be a deterministic function F, such that $F_\omega(x) = F(x)$ for every $x \in \mathbb{R}$ and a.e. $\omega \in \Omega$. Let $f(x) = F'(x)$, then f is independent of ω and we have that for a.e. $\omega \in \Omega$, $f = f_\omega$ as elements of $L^1(\mathbb{R}, dE)$. In particular, the deterministic set

$$A \equiv \{E \in \mathbb{R} \,|\, F'(x) \text{ exists and is finite and strictly positive}\} \tag{4.2}$$

is P-almost surely an essential support of the absolutely continuous part of μ_ω.

It remains to show that for a.e. $\omega \in \Omega$, $\mu_{\omega,\mathrm{sing}}(A) = 0$. By Corollary 1.1.1, A is P-almost surely an essential support of the absolutely continuous part of μ_{ω,δ_1}. Thus, for a.e. $\omega \in \Omega$, A equals, up to a set of zero Lebesgue measure, to the set

$$A_{\omega,\delta_1} \equiv \{E \in \mathbb{R} \,|\, \langle \delta_1, (H_\omega - E - i0)^{-1}\delta_1\rangle \text{ exists and } 0 < \operatorname{Im}\langle \delta_1, (H_\omega - E - i0)^{-1}\delta_1\rangle < \infty\}\,. \tag{4.3}$$

For any fixed $\omega \in \Omega$ and a rank one perturbation of H_ω of the form $H_{\omega,\lambda} \equiv H_\omega + \lambda\langle \delta_1, \cdot\rangle\delta_1$, (i) of Proposition 2.2 implies that $\mu_{\omega,\lambda,\delta_1,\mathrm{sing}}(A_{\omega,\delta_1}) = 0$ for any $\lambda \in \mathbb{R}$ (where $\mu_{\omega,\lambda,\delta_1,\mathrm{sing}}$ is the singular part of the spectral measure for $H_{\omega,\lambda}$ and δ_1). By (iii) of Proposition 2.2, this implies that $\mu_{\omega,\lambda,\delta_1,\mathrm{sing}}(A) = 0$ for Lebesgue a.e. $\lambda \in \mathbb{R}$. By the same kind of argument as in the proof of Corollary 1.1.2 (namely, due to the fact that the conditional probability distribution of $\lambda_\omega(1)$, given any $\{\lambda_\omega(m)\}_{m>1}$, is absolutely continuous w.r.t. Lebesgue measure), this implies that $\mu_{\omega,\delta_1,\mathrm{sing}}(A) = 0$ for a.e. $\omega \in \Omega$ (where $\mu_{\omega,\delta_1,\mathrm{sing}}$ is the singular part of the spectral measure for H_ω and δ_1). By Corollary 1.1.1, again, this implies that for a.e. $\omega \in \Omega$, $\mu_{\omega,\mathrm{sing}}(A) = 0$. $\square$

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