A note on the entropy production formula

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Dedicated to Professor Huzihiro Araki on the occasion of his 70th birthday

Abstract. We give an elementary derivation of the entropy production formula of [JP1] based on Araki Perturbation Theory of KMS states. Using this derivation we show that the entropy production of any normal, stationary state is zero.

1 Introduction

Let $\mathcal{O}$ be a C^*-algebra, $E(\mathcal{O})$ the set of all states on $\mathcal{O}$ and $\omega \in E(\mathcal{O})$. We assume that there exists a reference C^*-dynamics σ_ω^t on $\mathcal{O}$ such that ω is a $(\sigma_\omega, -1)$-KMS state. In typical applications, $\mathcal{O}$ will be the algebra of observables of a quantum system made of several components (for example reservoirs). Then, $\mathcal{O}$ will have a product structure and ω could be a product of KMS-states, possibly at different temperatures. We denote by δ_ω the generator of σ_ω^t (*i.e.* $\sigma_\omega^t = \mathrm{e}^{t\delta_\omega}$) and by $\mathcal{D}(\delta_\omega)$ its domain. Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS-representation of the algebra $\mathcal{O}$ associated to the state ω.

A state $\eta \in E(\mathcal{O})$ is called ω-normal if there exists a density matrix ρ_η on $\mathcal{H}_\omega$ such that, for all $A \in \mathcal{O}$, $\eta(A) = \mathrm{Tr}(\rho_\eta \pi_\omega(A))$. Let $\mathcal{N}_\omega$ be the set of all ω-normal states on $\mathcal{O}$.

For $\eta \in \mathcal{N}_\omega$, we denote by $\mathrm{Ent}(\eta|\omega)$ the relative entropy of Araki [Ar1, Ar2]. (We use the notational convention for relative entropy of [BR, Don].) If $\eta \notin \mathcal{N}_\omega$, we set $\mathrm{Ent}(\eta|\omega) = -\infty$. For unitary $U \in \mathcal{O}$ and $\eta \in E(\mathcal{O})$, we denote by η_U the state $\eta_U(A) \equiv \eta(U^* A U)$. The main result of this note is:

Theorem 1.1 *For any unitary $U \in \mathcal{O} \cap \mathcal{D}(\delta_\omega)$ and any $\eta \in E(\mathcal{O})$,*

$$\mathrm{Ent}(\eta_U|\omega) = \mathrm{Ent}(\eta|\omega) - \mathrm{i}\eta(U^* \delta_\omega(U)). \tag{1.1}$$

As we shall explain below, Theorem 1.1 is a natural generalization of the entropy production formula derived in [JP1, JP2]. The method of proof we will use in this note, however, is quite different from the one in [JP1]. We will reduce the proof of Theorem 1.1 to a fairly elementary application of some well known identities in Araki's theory of perturbation of KMS structure. The proof in [JP1], based on Araki-Connes cocycles, was technically more involved and restricted to *faithful* states $\eta \in \mathcal{N}_\omega$.

We now relate Equ. (1.1) to the entropy production formula of [JP1, JP2]. Assume that there exists a C^*-dynamics τ^t on $\mathcal{O}$ and that ω is τ-invariant. Let $V(t)$ be a time-dependent local perturbation, that is, $V(t)$ is norm-continuous, self-adjoint, $\mathcal{O}$-valued function on $\mathbb{R}$ (the time-independent case of [JP1] of course follows by setting $V(t) \equiv V$). The perturbed time evolution is the strongly continuous family of $*$-automorphisms of $\mathcal{O}$ given by the formula

$$\begin{aligned} \tau_V^t(A) &\equiv \tau^t(A) \\ &+ \sum_{n \geq 1} \mathrm{i}^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \, [\tau^{t_n}(V(t_n)), [\cdots, [\tau^{t_1}(V(t_1)), \tau^t(A)]]]. \end{aligned}$$

In the interaction representation, τ_V^t is given by

$$\tau_V^t(A) = \Gamma_V^t \tau^t(A) \Gamma_V^{t*},$$

where $\Gamma_V^t \in \mathcal{O}$ is a family of unitaries satisfying the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_V^t = \mathrm{i}\Gamma_V^t \tau^t(V(t)), \qquad \Gamma_V^0 = \mathbf{1}.$$

Theorem 1.1 then has the following immediate corollary (see also Theorem 4.8 in [JP2]):

Corollary 1.2 *Assume that ω is τ-invariant and that $\Gamma_V^t \in \mathcal{D}(\delta_\omega)$. Then, for any $\eta \in E(\mathcal{O})$,*

$$\mathrm{Ent}(\eta \circ \tau_V^t|\omega) = \mathrm{Ent}(\eta|\omega) - \mathrm{i}\eta(\Gamma_V^t \delta_\omega(\Gamma_V^{t*})). \tag{1.2}$$

From now on we will consider the time-independent case $V(t) \equiv V$. If $V \in \mathcal{D}(\delta_\omega)$, then $\Gamma_V^t \in \mathcal{D}(\delta_\omega)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_V^t \delta_\omega(\Gamma_V^{t*}) = -\mathrm{i}\tau_V^t(\delta_\omega(V)). \tag{1.3}$$

Hence, (1.2) reduces to the entropy production formula of [JP1]:

$$\mathrm{Ent}(\eta \circ \tau_V^t|\omega) = \mathrm{Ent}(\eta|\omega) - \int_0^t \eta \circ \tau_V^s(\delta_\omega(V))\,\mathrm{d}s. \tag{1.4}$$

We emphasize that the above derivation of (1.4) allows for non-faithful η.

The entropy production of a state $\eta \in E(\mathcal{O})$ was defined in [JP1, JP2] by $\mathrm{Ep}_V(\eta) \equiv \eta(\delta_\omega(V))$, see also [OHI, O1, O2, Ru, Sp]. On physical grounds, it is natural to conjecture that if η is ω-normal and τ_V-invariant, then $\mathrm{Ep_V}(\eta) = 0$. For faithful η this was proven in [JP1]. Here, we establish this result in full generality.

Theorem 1.3 *Assume that ω is τ-invariant, that $V \in \mathcal{D}(\delta_\omega)$ and that η is τ_V-invariant and ω-normal. Then,*

$$\mathrm{Ep}_V(\eta) = 0.$$

Remark. If $\mathrm{Ent}(\eta|\omega) > -\infty$, then Theorem 1.3 is an immediate consequence of Equ. (1.4). The case $\mathrm{Ent}(\eta|\omega) = -\infty$ requires a separate and somewhat delicate argument.

The results of this note were announced in the recent review [JP2] where the interested reader may find additional information and references about entropy production and its role in non-equilibrium quantum statistical mechanics.

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2 Proof of Theorem 1.1

We assume that the reader is familiar with basic results of Tomita-Takesaki modular theory as discussed, for example, in [BR, DJP, Don, OP].

Let $\mathfrak{M}_\omega \equiv \pi_\omega(\mathcal{O})''$ be the enveloping von Neumann algebra. Since ω is $(\sigma_\omega, -1)$-KMS state, the vector Ω_ω is separating for $\mathfrak{M}_\omega$, and we denote by $\mathcal{P}$, J, Δ_ω the corresponding natural cone, modular conjugation and modular operator. We recall that $\Delta_\omega = \mathrm{e}^{L_\omega}$, where L_ω is the unique self-adjoint operator on $\mathcal{H}_\omega$ such that

$$\pi_\omega(\sigma_\omega^t(A)) = \mathrm{e}^{\mathrm{i}tL_\omega}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL_\omega}, \qquad L_\omega\Omega_\omega = 0.$$

In particular, σ_ω^t extends naturally to a W^*-dynamics on $\mathfrak{M}_\omega$ which we again denote by σ_ω^t. In this context σ_ω^t is called modular dynamics.

Any state $\eta \in \mathcal{N}_\omega$ has a unique normal extension to $\mathfrak{M}_\omega$ which we denote by the same letter. Obviously, η is ω-normal iff η_U is ω-normal for all unitaries $U \in \mathcal{O}$ and so, in the proof of Theorem 1.1, we may restrict ourselves to ω-normal η's.

We will use the fact that if $\gamma : \mathfrak{M}_\omega \mapsto \mathfrak{M}_\omega$ is a $*$-automorphism, then

$$\mathrm{Ent}(\eta \circ \gamma | \omega \circ \gamma) = \mathrm{Ent}(\eta|\omega).$$

In particular,

$$\mathrm{Ent}(\eta_U|\omega) = \mathrm{Ent}(\eta|\omega_{U^*}).$$

Let Ψ_{U^*} be the unique vector representative of the state ω_{U^*} in the cone $\mathcal{P}$. A simple computation shows that

$$\Psi_{U^*} = \pi_\omega(U^*) J \pi_\omega(U^*) \Omega_\omega.$$

We will consider $P \equiv \pi_\omega(-\mathrm{i}U^*\delta_\omega(U))$ as a local perturbation of the modular group σ_ω^t. Let α^t be the locally perturbed W^*-dynamics,

$$\alpha^t(A) \equiv \mathrm{e}^{\mathrm{i}t(L_\omega+P)} A \mathrm{e}^{-\mathrm{i}t(L_\omega+P)} = \Theta_P^t \sigma_\omega^t(A) \Theta_P^{t*},$$

where $\mathrm{e}^{\mathrm{i}t(L_\omega+P)}\mathrm{e}^{-\mathrm{i}tL_\omega} \equiv \Theta_P^t \in \mathfrak{M}_\omega$ is a family of unitaries satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_P^t = \mathrm{i}\Theta_P^t \sigma_\omega^t(P), \qquad \Theta_P^0 = \mathbf{1}. \tag{2.5}$$

Let ψ be the unique $(\alpha, -1)$-KMS state on $\mathfrak{M}_\omega$. By the Araki theory, $\Omega_\omega \in \mathcal{D}(\mathrm{e}^{(L_\omega+P)/2})$ and the unique vector representative of ψ in the natural cone $\mathcal{P}$ is

$$\Psi = \frac{\mathrm{e}^{(L_\omega+P)/2}\Omega_\omega}{\|\mathrm{e}^{(L_\omega+P)/2}\Omega_\omega\|}.$$

Another fundamental result of Araki's theory is the relation

$$\mathrm{Ent}(\eta|\psi) = \mathrm{Ent}(\eta|\omega) + \eta(P) - \log \|\mathrm{e}^{(L_\omega+P)/2}\Omega_\omega\|^2, \tag{2.6}$$

which holds for all ω-normal states η. (For η faithful, this relation was proven in [Ar1, Ar2], see also [BR]. Its extension to general η was obtained in [Don], see also the next section). Hence, to finish the proof it suffices to show that $\mathrm{e}^{(L_\omega+P)/2}\Omega_\omega = \Psi_{U^*}$.

We set $T^t \equiv U^*\sigma_\omega^t(U)$ and observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}T^t = \mathrm{i}T^t\sigma_\omega^t(-\mathrm{i}U^*\delta_\omega(U)), \qquad T^0 = \mathbf{1}.$$

Comparison with Equ. (2.5) immediately leads to $\pi_\omega(T^t) = \Theta_P^t$ and therefore

$$\begin{aligned} e^{it(L_\omega+P)}\Omega_\omega &= \pi_\omega(T^t)e^{itL_\omega}\Omega_\omega \\ &= \pi_\omega(U^*)e^{itL_\omega}\pi_\omega(U)\Omega_\omega. \end{aligned} \tag{2.7}$$

Since the vector-valued function $z \mapsto e^{iz(L_\omega+P)}\Omega_\omega$ is analytic inside the strip $-1/2 < \operatorname{Im} z < 0$ and strongly continuous on its closure, analytic continuation of the identity (2.7) to $z = -i/2$, yields

$$\begin{aligned} e^{(L_\omega+P)/2}\Omega_\omega &= \pi_\omega(U^*)\Delta_\omega^{1/2}\pi_\omega(U)\Omega_\omega \\ &= \pi_\omega(U^*)J\pi_\omega(U^*)\Omega_\omega \\ &= \Psi_{U^*}, \end{aligned}$$

which is the desired relation.

3 Proof of Theorem 1.3

Let Ω_η be the vector representative of η in the natural cone $\mathcal{P}$. The standard Liouvillean associated to the dynamics τ_V^t is $L_V = L + \pi_\omega(V) - J\pi_\omega(V)J$, where L is the standard Liouvillean associated to τ^t. We recall that L and L_V are uniquely specified by

$$\pi_\omega(\tau^t(A)) = e^{itL}\pi_\omega(A)e^{-itL}, \qquad L\Omega_\omega = 0,$$

and

$$\pi_\omega(\tau_V^t(A)) = e^{itL_V}\pi_\omega(A)e^{-itL_V}, \qquad L_V\Omega_\eta = 0.$$

We denote by s_η the support of the state η and set $s'_\eta = Js_\eta J$. Obviously

$$s_\eta\Omega_\eta = s'_\eta\Omega_\eta = \Omega_\eta,$$

and since η is τ_V-invariant

$$e^{itL_V}s_\eta = s_\eta e^{itL_V}, \qquad e^{itL_V}s'_\eta = s'_\eta e^{itL_V}.$$

Let $\Delta_{\omega|\eta}$ be the relative modular operator. We recall that $\operatorname{Ker}\Delta_{\omega|\eta} = \operatorname{Ker} s'_\eta$,

$$J\Delta_{\omega|\eta}^{1/2}A\Omega_\eta = s'_\eta A^*\Omega_\omega,$$

for all $A \in \mathfrak{M}_\omega$ and that $\Delta_{\omega|\eta}$ is essentially self-adjoint on $\mathfrak{M}_\omega\Omega_\eta + (\mathbf{1} - s'_\eta)\mathcal{H}_\omega$. Hence

$$\Delta_{\omega\circ\tau_V^t|\eta\circ\tau_V^t} = e^{-itL_V}\Delta_{\omega|\eta}e^{itL_V},$$

and since η is τ_V-invariant,

$$\mathrm{e}^{\mathrm{i}tL_V}\Delta_{\omega|\eta}\mathrm{e}^{-\mathrm{i}tL_V} = \Delta_{\omega\circ\tau_V^{-t}|\eta} = \Delta_{\omega_{U^*}|\eta},$$

where $U^* \equiv \Gamma_V^t$.

As in the proof of Theorem 1.1, we set $P \equiv \pi_\omega(-\mathrm{i}U^*\delta_\omega(U))$ and denote by α the perturbation of the modular dynamics σ_ω by P. It follows that ω_{U^*} is the unique $(\alpha, -1)$-KMS state. Since also $\|\mathrm{e}^{(L_\omega+P)/2}\Omega_\omega\| = 1$, the basic perturbation formula of Araki-Donald (see Lemma 5.7 in [Don]) yields

$$s'_\eta \log \Delta_{\omega_{U^*}|\eta} = s'_\eta \log \Delta_{\omega|\eta} - s'_\eta P.$$

Hence,

$$\mathrm{e}^{\mathrm{i}tL_V} s'_\eta \log \Delta_{\omega|\eta}\mathrm{e}^{-\mathrm{i}tL_V} = s'_\eta \log \Delta_{\omega|\eta} - s'_\eta P,$$

and we conclude that for any real number $\lambda \neq 0$,

$$\mathrm{e}^{\mathrm{i}tL_V}\left(s'_\eta \log \Delta_{\omega|\eta} + \mathrm{i}\lambda\right)^{-1}\mathrm{e}^{-\mathrm{i}tL_V} = \left(s'_\eta \log \Delta_{\omega|\eta} - s'_\eta P + \mathrm{i}\lambda\right)^{-1}.$$

Since $\mathrm{e}^{-\mathrm{i}tL_V}\Omega_\eta = \Omega_\eta$, the second resolvent identity yields that for all real $\lambda \neq 0$,

$$(\Omega_\eta, (s'_\eta \log \Delta_{\omega|\eta} + \mathrm{i}\lambda)^{-1} s'_\eta P (s'_\eta \log \Delta_{\omega|\eta} - s'_\eta P + \mathrm{i}\lambda)^{-1}\Omega_\eta) = 0.$$

Since

$$\mathrm{s}-\lim_{\lambda\to\infty} \mathrm{i}\lambda\left(s'_\eta \log \Delta_{\omega|\eta} + \mathrm{i}\lambda\right)^{-1} = \mathbf{1},$$

and

$$\mathrm{s}-\lim_{\lambda\to\infty} \mathrm{i}\lambda\left(s'_\eta \log \Delta_{\omega|\eta} - s'_\eta P + \mathrm{i}\lambda\right)^{-1} = \mathbf{1},$$

we derive that

$$(\Omega_\eta, P\Omega_\eta) = (\Omega_\eta, s'_\eta P\Omega_\eta) = 0.$$

On the other hand, using Equ. (1.3), we get

$$P = \pi_\omega(-\mathrm{i}U^*\delta_\omega(U)) = -\int_0^t \pi_\omega(\tau_V^s(\delta_\omega(V)))\,\mathrm{d}s,$$

and since η is τ_V-invariant we conclude that

$$0 = (\Omega_\eta, P\Omega_\eta) = -\int_0^t \eta\circ\tau_V^s(\delta_\omega(V))\,\mathrm{d}s = -t\eta(\delta_\omega(V)),$$

for all t. This yields the statement.

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