A non-commutative Lévy-Cramér continuity theorem

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September 4, 2008

Abstract

The classical Lévy-Cramér continuity theorem asserts that the convergence of the characteristic functions implies the weak convergence of the corresponding probability measures. We extend this result to the setting of non-commutative probability theory and discuss some applications.

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1 Introduction

In classical probability, the Lévy-Cramér continuity theorem is a standard tool for proving convergence in distribution of a family of random variables. To recall its statement, let $\mathbb{T}$ denote either $\mathbb{N}$ or $\mathbb{R}$, $\overline{\mathbb{T}} \equiv \mathbb{T} \cup \{\infty\}$, and let $x \cdot y$ be the standard inner product of two vectors $x, y \in \mathbb{R}^n$. Suppose that, for each $t \in \overline{\mathbb{T}}$, $A_t = (A_t^{(1)}, \ldots, A_t^{(n)})$ is a $\mathbb{R}^n$-valued random variable on the probability space $(\Omega_t, \mathcal{F}_t, \mathbb{P}_t)$ and denote by $\mathbb{E}_t$ the expectation with respect to $\mathbb{P}_t$. The Lévy-Cramér continuity theorem asserts that if

$$\lim_{t\to\infty} \mathbb{E}_t(\mathrm{e}^{\mathrm{i}\alpha\cdot A_t}) = \mathbb{E}_\infty(\mathrm{e}^{\mathrm{i}\alpha\cdot A_\infty}), \tag{1}$$

for all $\alpha \in \mathbb{R}^n$, then A_t converges to A_∞ in distribution, *i.e.,* for every bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$

$$\lim_{t\to\infty} \mathbb{E}_t(f(A_t)) = \mathbb{E}_\infty(f(A_\infty)). \tag{2}$$

More generally, let $\mathcal{D}(f)$ be the set of discontinuity points of a function f. Then (2) holds for every bounded Borel function f such that

$$\mathbb{P}_\infty([A_\infty \in \mathcal{D}(f)]) = 0,$$

see Theorem 29.2 in [Bi].

We are interested in non-commutative analogues of these results. To discuss such generalizations we need to briefly recall some basic notions of non-commutative probability theory. We refer to [Mey] or [Maa] for a more detailed introduction and to [BR1] for the theory of von Neumann algebras.

We start with an algebraic reformulation of classical (commutative) probability theory. A bounded, real-valued random variable A on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ can be identified with a real element of the set $\mathfrak{M} \equiv \mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ of equivalence classes of essentially bounded $\mathcal{F}$-measurable functions on Ω. An event $E \in \mathcal{F}$ (or rather, an equivalence class under the equivalence $A \sim B \Leftrightarrow \mathbb{P}(A\Delta B) = 0$) can be identified with the random variable $\mathbb{1}_E \in \mathfrak{M}$, which satisfies $\mathbb{1}_E^2 = \overline{\mathbb{1}_E} = \mathbb{1}_E$; conversely, any element $A \in \mathfrak{M}$ satisfying $A^2 = \bar{A} = A$ is the equivalence class of the indicator function of some set $E \in \mathcal{F}$. Denoting by $\mathbb{E}$ the expectation associated with $\mathbb{P}$, the law of a random variable A is defined as the unique probability measure μ on $\mathbb{R}$ such that $\mathbb{E}(f(A)) = \int f(x)\,\mathrm{d}\mu(x)$ for all bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$.

Note that $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is a commutative von Neumann algebra. Its elements can be interpreted as bounded multiplication operators on the Hilbert space $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. In non-commutative probability theory $\mathfrak{M}$ becomes a general von Neumann algebra (weakly closed $*$-subalgebras of $\mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on some Hilbert space $\mathcal{H}$). In this extended framework, a random variable is an element $A \in \mathfrak{M}$ satisfying $A = A^*$, *i.e.,* a selfadjoint operator of $\mathcal{H}$. An event is an element $A \in \mathfrak{M}$ satisfying $A^2 = A^* = A$, *i.e.,* the orthogonal projection on a closed subspace of $\mathcal{H}$. The role of the expectation is played by a normal state ω on $\mathfrak{M}$, *i.e.,* a positive linear functional on $\mathfrak{M}$ ($\omega(B^*B) \geq 0$ for all $B \in \mathfrak{M}$) which is continuous under monotone convergence and normalized by the condition $\omega(I) = 1$. The law of A in the state ω is the unique measure ω^A on $\mathbb{R}$ such that $\omega(f(A)) = \int f(x)\,\mathrm{d}\omega^A(x)$. The existence of such a measure follows from the von Neumann spectral theorem (see Theorem VIII.6 in [RS]): there exists a projection valued spectral measure ξ^A on $\mathbb{R}$, with support on the spectrum $\mathrm{Sp}\,A$ of A, such that $(u, Au) = \int_{\mathrm{Sp}\,A} x\,\mathrm{d}(u, \xi^A(x)u)$ for all $u \in \mathcal{H}$. For every bounded Borel function f and $u \in \mathcal{H}$ one has $(u, f(A)u) = \int_{\mathrm{Sp}\,A} f(x)\,\mathrm{d}(u, \xi^A(x)u)$. In particular, $\omega \circ \xi^A$ is a probability measure,

$$\omega(f(A)) = \int_{\mathrm{Sp}\,A} f(x)\,\mathrm{d}(\omega \circ \xi^A)(x), \tag{3}$$

and so $\omega^A = \omega \circ \xi^A$ is the law of A. Clearly, $\mathrm{supp}\,\omega^A \subset \mathrm{Sp}\,A$. If ω is faithful, then $\mathrm{supp}\,\omega^A = \mathrm{Sp}\,A$; otherwise this may not be the case. Note that the framework thus defined extends the classical one: as already remarked, the space $\mathrm{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$, acting by multiplication on the Hilbert space $\mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$.

As long as one considers the law of one single random variable at a time, non-commutative probability reduces to classical probability. For example, within the framework of classical probability one can discuss the convergence in distribution of a sequence of non-commuting random variables in a given state. The novel aspects of non-commutative probability emerge only when one considers simultaneously two (or more) *non-commuting* random variables A,$B \in \mathfrak{M}$. Then, it is in general impossible to define a joint law for A and B: there is no measure μ on $\mathbb{R}^2$ such that

$$\omega(f(A)g(B)) = \int f(x)g(y)\,\mathrm{d}\mu(x, y),$$

for all bounded continuous functions f and g. In particular there is no measure μ such that

$$\omega(\mathrm{e}^{\mathrm{i}\alpha A}\mathrm{e}^{\mathrm{i}\beta B}) = \int \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{\mathrm{i}\beta y}\,\mathrm{d}\mu(x,y),$$

for all $\alpha, \beta \in \mathbb{R}$. For this reason, quantities such as

$$\omega(\mathrm{e}^{\mathrm{i}\alpha_1 A^{(1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_n A^{(n)}}),$$

which we call *quasi-characteristic functions* in accordance with [CH], do not have a direct probabilistic interpretation. In particular, an assumption analogous to (1),

$$\lim_{t\to\infty} \omega_t(\mathrm{e}^{\mathrm{i}\alpha_1 A_t^{(1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_n A_t^{(n)}}) = \omega_\infty(\mathrm{e}^{\mathrm{i}\alpha_1 A_\infty^{(1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_n A_\infty^{(n)}}), \tag{4}$$

where $A_t^{(i)}$ are non-commuting self-adjoint elements of some von Neumann algebra $\mathfrak{M}_t$, cannot be interpreted as a convergence of measures because, in general, neither the finite t quantities, nor their limit, are characteristic functions of probability measures. Assumptions such as (4) were often considered in the non-commutative probability literature, but their rigorous implications were rarely studied (the only two exceptions we are aware of are [CH] and its extension [CGH], and [Kup], see subsection 4.2). Instead, it was generally considered that such a convergence was a good indication of the relevance of the limiting structure $(\mathfrak{M}_\infty, \omega_\infty)$ (another commonly used approach with similar motivations uses moments, see in particular [GvW], [AB]).

Assuming that (4) holds (see Assumption (A) below for the precise formulation of our main condition), we shall prove in this paper that the relation

$$\lim_{t\to\infty} \omega_t(f_1(A_t^{(1)}) \cdots f_n(A_t^{(n)})) = \omega_\infty(f_1(A_\infty^{(1)}) \cdots f_n(A_\infty^{(n)})) \tag{5}$$

holds for all continuous functions $f_1, \ldots, f_n$. As in the classical case, an extension to discontinuous functions exists, but under assumptions stronger than those one might naively expect. Although in the non commutative probability the relation (5) has no *bona fide* measure theoretic interpretation, it is relevant in the theory of repeated measurement of quantum systems (see [Dav]). From the mathematical point of view, we consider the implications of the type (4) $\Rightarrow$ (5) a natural non-commutative extension of the classical Lévy-Cramér continuity theorem.

The paper is organized as follows. Our main results are stated in section 2 and the proofs are given in Section 3. Section 4 is devoted to discussion of our results including examples, applications and comparison with the literature.

Acknowledgment. A part of this work has been done during Y.P.'s visit to McGill University funded by NSERC. The research of V.J. was partly supported by NSERC.

2 Non-commutative Lévy-Cramér continuity theorems

Recall that $\mathbb{T}$ denotes either $\mathbb{N}$ or $\mathbb{R}$ and $\overline{\mathbb{T}} \equiv \mathbb{T} \cup \{\infty\}$. For any $t \in \overline{\mathbb{T}}$ let

(i) $\mathfrak{M}_t$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}_t$;

(ii) ω_t be a normal state on $\mathfrak{M}_t$;

(iii) $A_t^{(1)}, \ldots, A_t^{(n)}$ be (possibly unbounded) selfadjoint operators on $\mathcal{H}_t$ which are affiliated to $\mathfrak{M}_t$, *i.e.*, such that $\mathrm{e}^{\mathrm{i}\alpha A_t^{(j)}} \in \mathfrak{M}_t$ for all $\alpha \in \mathbb{R}$.

$\mathcal{C}$ denotes the set of real bounded continuous functions on $\mathbb{R}$, $\mathcal{M}$ the set of Borel probability measures on $\mathbb{R}$ and $\mathcal{B}$ the set of real bounded Borel functions on $\mathbb{R}$. For $f \in \mathcal{B}$ and $\mu \in \mathcal{M}$, $\mathcal{D}(f)$ denotes the set of discontinuity points of f ($\mathcal{D}(f)$ is Borel, see *e.g.*, Theorem 25.7 in [Bi]) and $\mu(f)$ denotes $\int f\,\mathrm{d}\mu$. Finally, $\omega_t^{(j)}$ denotes the law of $A_t^{(j)}$ in the state ω_t, *i.e.*, the unique element of $\mathcal{M}$ satisfying

$$\omega_t^{(j)}(f) = \omega_t(f(A_t^{(j)})),$$

for all $f \in \mathcal{C}$.

Our central assumption is:

Assumption (A) For all $\alpha \in \mathbb{R}^m$, $j_1, \ldots, j_m \in \{1, \ldots, n\}$ with $m \geq 1$, one has

$$\lim_{t \to \infty} \omega_t \left(\mathrm{e}^{\mathrm{i}\alpha_1 A_t^{(j_1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_m A_t^{(j_m)}} \right) = \omega_\infty \left(\mathrm{e}^{\mathrm{i}\alpha_1 A_\infty^{(j_1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_m A_\infty^{(j_m)}} \right).$$

2.1 Statement of the results

Our first result is the following non-commutative version of the Lévy-Cramér continuity theorem.

Theorem 1 *Under Assumption (A),*

$$\lim_{t \to \infty} \omega_t \left(f_1(A_t^{(1)}) \cdots f_n(A_t^{(n)}) \right) = \omega_\infty \left(f_1(A_\infty^{(1)}) \cdots f_n(A_\infty^{(n)}) \right), \tag{6}$$

holds for all $f_1, \ldots, f_n \in \mathcal{C}$.

This result can be extended to bounded Borel functions as follows.

Theorem 2 *Under Assumption (A) there exists a family $\mathfrak{S} = (S_j)_{j \in \{1, \ldots, n\}}$ of subsets of $\mathcal{M}$ such that*

$$\lim_{t \to \infty} \omega_t \left(f_1(A_t^{(1)}) \cdots f_n(A_t^{(n)}) \right) = \omega_\infty \left(f_1(A_\infty^{(1)}) \cdots f_n(A_\infty^{(n)}) \right), \tag{7}$$

holds if, for all $j \in \{1, \ldots, n\}$, $f_j \in \mathcal{B}$ and $\sigma(\mathcal{D}(f_j)) = 0$ for every $\sigma \in S_j$.

We shall say that a family $\mathfrak{S} = (S_j)_{j \in \{1, \ldots, n\}}$ of subsets of $\mathcal{M}$ is admissible if (7) holds under the conditions of Theorem 2.

Remarks. 1. In general, the choice of $\mathfrak{S}$ is not unique and the subsets $S_j \subset \mathcal{M}$ for different j can not be chosen independently of one another. Explicit examples of admissible families are given in Subsection 2.2.
2. We will see that possible choices for $\mathfrak{S}$ imply a strengthening of the continuity assumption with respect to the classical Lévy-Cramér theorem. This strengthening is necessary and due to the non commutativity of the problem at hand. We illustrate this in subsection 4.1.

In the case where ω_∞ is faithful on the algebra $\mathfrak{M}_\infty$, Lemma 7 below shows that $S_j = \{\omega_\infty^{(j)}\}$ defines an admissible family. Theorem 2 then yields an optimal non-commutative extension of the classical Lévy-Cramér theorem.

Corollary 3 *If Assumption (A) holds and ω_∞ is faithful on $\mathfrak{M}_\infty$ then (7) holds for $f_1, \ldots, f_n \in \mathcal{B}$ satisfying $\omega_\infty^{(j)}(\mathcal{D}(f_j)) = 0$ for every $j \in \{1, \ldots, n\}$.*

2.2 Admissible families

In this subsection we introduce possible choices of admissible families. We then discuss the special case where ω_∞ is faithful. Note that if ω is a normal state on the von Neumann algebra $\mathfrak{M}$ then, for any unitary $U \in \mathfrak{M}$, the formula $\omega_U(\,\cdot\,) \equiv \omega(U^*(\,\cdot\,)U)$ defines a normal state on $\mathfrak{M}$. In particular, for $t \in \overline{\mathbb{T}}$, $j \in \{1, \ldots, n\}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ we can define the following normal states on $\mathfrak{M}_t$,

$$\omega_{jt}^-(\alpha_1, \ldots, \alpha_{j-1};\,\cdot\,) \equiv \omega_t \left(\mathrm{e}^{\mathrm{i}\alpha_1 A_t^{(1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_{j-1} A_t^{(j-1)}} (\,\cdot\,) \mathrm{e}^{-\mathrm{i}\alpha_{j-1} A_t^{(j-1)}} \cdots \mathrm{e}^{-\mathrm{i}\alpha_1 A_t^{(1)}} \right),$$

$$\omega_{jt}^+(\alpha_{j+1}, \ldots, \alpha_n;\,\cdot\,) \equiv \omega_t \left(\mathrm{e}^{-\mathrm{i}\alpha_n A_t^{(n)}} \cdots \mathrm{e}^{-\mathrm{i}\alpha_{j+1} A_t^{(j+1)}} (\,\cdot\,) \mathrm{e}^{\mathrm{i}\alpha_{j+1} A_t^{(j+1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_n A_t^{(n)}} \right).$$

Definition 4 *By Equ. (3), the maps*

$$\alpha \mapsto \omega_{jt}^- \left(\alpha_1, \ldots, \alpha_{j-1}; \mathrm{e}^{\mathrm{i}\alpha A_t^{(j)}} \right),$$

$$\alpha \mapsto \omega_{jt}^+ \left(\alpha_{j+1}, \ldots, \alpha_n; \mathrm{e}^{\mathrm{i}\alpha A_t^{(j)}} \right),$$

are characteristic functions of probability laws that we denote by $\sigma_{jt}^-(\alpha_1, \ldots, \alpha_{j-1};\,\cdot\,)$ and $\sigma_{jt}^+(\alpha_{j+1}, \ldots, \alpha_n;\,\cdot\,)$ respectively.

Note in particular that $\sigma_{1t}^- = \omega_t^{(1)}$ and $\sigma_{nt}^+ = \omega_t^{(n)}$. We define

$$\begin{aligned} S_j^- &\equiv \{\sigma_{j\infty}^-(\alpha_1,\dots,\alpha_{j-1}) \,|\, \alpha_1,\dots,\alpha_{j-1}\in\mathbb{R}\}, \\ S_j^+ &\equiv \{\sigma_{j\infty}^+(\alpha_{j+1},\dots,\alpha_n) \,|\, \alpha_{j+1},\dots,\alpha_n\in\mathbb{R}\}, \end{aligned} \tag{8}$$

for $j\in\{1,\dots,n\}$.

We can now define possible choices of admissible families.

Theorem 5 *For any $J\in\{0,\dots,n\}$ the family $(S_j)_{j\in\{1,\dots,n\}}$ defined by*

$$S_j \equiv \begin{cases} S_j^- & \text{if } j\le J, \\ S_j^+ & \text{if } j>J. \end{cases} \tag{9}$$

is admissible.

The reason for the multiplicity of choices of admissible families will become clear in Subsection 3.2.

The following continuity properties of the maps $t\mapsto\sigma_{jt}^{\pm}$ are an easy consequences of the classical Lévy-Cramér continuity theorem.

Lemma 6 *Fix $j\in\{1,\dots,n\}$ and let $g\in\mathcal{B}$. If*

$$\sigma(\mathcal{D}(g)) = 0 \text{ for every } \sigma\in S_j^-, \tag{10}$$

then

$$\lim_{t\to\infty}\sigma_{jt}^-(\alpha_1,\dots,\alpha_{j-1};g) = \sigma_{j\infty}^-(\alpha_1,\dots,\alpha_{j-1};g), \tag{11}$$

for every $\alpha_1,\dots,\alpha_{j-1}\in\mathbb{R}$. Similarly, if

$$\sigma(\mathcal{D}(g)) = 0 \text{ for every } \sigma\in S_j^+, \tag{12}$$

then

$$\lim_{t\to\infty}\sigma_{jt}^+(\alpha_{j+1},\dots,\alpha_n;g) = \sigma_{j\infty}^+(\alpha_{j+1},\dots,\alpha_n;g), \tag{13}$$

for every $\alpha_{j+1},\dots,\alpha_n\in\mathbb{R}$. Finally, if

$$\omega_\infty^{(j)}(\mathcal{D}(g)) = 0, \tag{14}$$

then

$$\lim_{t\to\infty}\omega_t^{(j)}(g) = \omega_\infty^{(j)}(g). \tag{15}$$

Note that $\omega_\infty^{(j)}\in S_j^-\cap S_j^+$ and so (14) is a weaker assumption than (10) or (12). The following lemma shows that they are equivalent in the case where ω_∞ is faithful.

Lemma 7 *If ω_∞ is faithful on $\mathfrak{M}_\infty$, then for all $j\in\{1,\dots,n\}$, any $\sigma\in S_j^+\cup S_j^-$ is equivalent to $\omega_\infty^{(j)}$ (i.e., σ and $\omega_\infty^{(j)}$ are mutually absolutely continuous).*

Lemmas 6 and 7 are proven in Subsections 3.3 and 3.4. Theorems 2 and 5 are proven in Subsection 3.2.

3 Proofs

We will first prove Theorem 1 for a restricted class of bounded continuous functions. The result will then be extended to bounded Borel functions using an approximation procedure and Lemma 6.

3.1 Approximation of bounded Borel functions

Let $\mathcal{F} \subset \mathcal{C}$ denote the set of functions of the form

$$f(a) = \int_{\mathbb{R}} \hat{f}(\alpha)\, \mathrm{e}^{\mathrm{i}a\alpha}\, \mathrm{d}\alpha,$$

where $\hat{f} \in \mathrm{L}^1(\mathbb{R})$.

Lemma 8 *The conclusion* (6) *of Theorem 1 holds for any* $f_1, \dots, f_n \in \mathcal{F}$.

Proof. For any $j \in \{1, \dots, n\}$, $t \in \overline{\mathbb{T}}$ and $u, v \in \mathcal{H}_t$ it follows from the functional calculus that

$$(u, f_j(A_t^{(j)})v) = \int \hat{f}_j(\alpha)(u, \mathrm{e}^{\mathrm{i}\alpha A_t^{(j)}} v)\, \mathrm{d}\alpha.$$

The σ-weak continuity of ω_t thus allows us to conclude that

$$\omega_t(B f_j(A_t^{(j)}) C) = \int \hat{f}_j(\alpha)\, \omega_t(B \mathrm{e}^{\mathrm{i}\alpha A_t^{(j)}} C)\, \mathrm{d}\alpha,$$

for any $B, C \in \mathfrak{M}_t$. Invoking Fubini's theorem, one easily concludes that

$$\omega_t\left(f_1(A_t^{(1)}) \cdots f_n(A_t^{(n)})\right) = \int \hat{f}_1(\alpha_1) \cdots \hat{f}_n(\alpha_n)\, \omega_t\left(\mathrm{e}^{\mathrm{i}\alpha_1 A_t^{(1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_n A_t^{(n)}}\right) \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n,$$

for any $t \in \overline{\mathbb{T}}$. The claim then follows from Assumption (A) and Lebesgue's dominated convergence theorem.□

Lemma 9 *For any* $f \in \mathcal{B}$ *such that* $\sup_{a\in\mathbb{R}} |f(a)| \leq R$ *there exists a sequence* $(f_k)_{k\in\mathbb{N}}$ *in* $\mathcal{F}$ *such that*

$$\sup_{k\in\mathbb{N}, a\in\mathbb{R}} |f_k(a)| \leq R,$$

and

$$\lim_k f_k(a) = f(a),$$

for all $a \in \mathbb{R} \setminus \mathcal{D}(f)$.

Proof. For $k \in \mathbb{N}$ set

$$\hat{f}_k(\alpha) = \mathrm{e}^{-\alpha^2/2(k+1)} \int_{-k}^{+k} f(a)\, \mathrm{e}^{-\mathrm{i}a\alpha} \frac{\mathrm{d}a}{2\pi},$$

and notice that $|\hat{f}_k(\alpha)| \leq \mathrm{e}^{-\alpha^2/2(k+1)} kR/\pi \in \mathrm{L}^1(\mathbb{R})$. The Fourier transform of $\hat{f}_k$ can be written as

$$f_k(a) = \int_{\mathbb{R}} \mathbb{1}_{[-1,1]}\left(\frac{a}{k} + \frac{b}{k^{3/2}}\right) f\left(a + \frac{b}{k}\right) \mathrm{d}\nu(b),$$

where ν is the centered Gaussian measure of variance 1. It immediately follows that $\sup_{a\in\mathbb{R}} |f_k(a)| \leq R$. For $a \in \mathbb{R} \setminus \mathcal{D}(f)$, Lebesgue's dominated convergence theorem and the fact that $\lim_k f(a + b/k) = f(a)$ for all $b \in \mathbb{R}$ imply $\lim_k f_k(a) = f(a)$. □

3.2 Proof of Theorems 2 and 5

Let $f_1, \dots, f_n \in \mathcal{B}$ and set $R \equiv \max_j(1 + \sup_{a\in\mathbb{R}} |f_j(a)|)$. Fix $J \in \{0, \dots, n\}$ and define S_j according to (9). Denote by $(f_{j,k})_{k\in\mathbb{N}} \subset \mathcal{F}$ the approximating sequence for f_j given by Lemma 9. Writing

$$\begin{aligned}
\Delta_t &\equiv \omega_t\left(f_1(A_t^{(1)}) \cdots f_n(A_t^{(n)})\right) - \omega_\infty\left(f_1(A_\infty^{(1)}) \cdots f_n(A_\infty^{(n)})\right) \\
&= \omega_t\left(f_1(A_t^{(1)}) \cdots f_n(A_t^{(n)}) - f_{1,k_1}(A_t^{(1)}) \cdots f_{n,k_n}(A_t^{(n)})\right) \\
&+ \omega_t\left(f_{1,k_1}(A_t^{(1)}) \cdots f_{n,k_n}(A_t^{(n)})\right) - \omega_\infty\left(f_{1,k_1}(A_\infty^{(1)}) \cdots f_{n,k_n}(A_\infty^{(n)})\right) \\
&+ \omega_\infty\left(f_{1,k_1}(A_\infty^{(1)}) \cdots f_{n,k_n}(A_\infty^{(n)}) - f_1(A_\infty^{(1)}) \cdots f_n(A_\infty^{(n)})\right),
\end{aligned}$$

and applying Lemma 8 we get

$$\limsup_{t\to\infty} |\Delta_t| \le \limsup_{t\to\infty} \left|\omega_t \left(f_1(A_t^{(1)}) \cdots f_n(A_t^{(n)}) - f_{1,k_1}(A_t^{(1)}) \cdots f_{n,k_n}(A_t^{(n)}) \right)\right| + \left|\omega_\infty \left(f_1(A_\infty^{(1)}) \cdots f_n(A_\infty^{(n)}) - f_{1,k_1}(A_\infty^{(1)}) \cdots f_{n,k_n}(A_\infty^{(n)}) \right)\right|,$$

for any $k_1, \ldots, k_n \in \mathbb{N}$. To study the right hand side of this inequality we fix $s \in \overline{\mathbb{T}}$, set $F_j \equiv f_j(A_s^{(j)})$, $G_j \equiv f_{j,k_j}(A_s^{(j)})$ and proceed with the algebraic identity

$$F_1 \cdots F_n - G_1 \cdots G_n = \sum_{j=1}^{J} G_1 \cdots G_{j-1}(F_j - G_j)F_{j+1} \cdots F_n + \sum_{j=J+1}^{n} G_1 \cdots G_J F_{J+1} \cdots F_{j-1}(F_j - G_j)G_{j+1} \cdots G_n. \tag{16}$$

The terms of the first sum on the right hand side of this identity can be estimated as follows. Starting from the Fourier representation (see the proof of Lemma 8)

$$\omega_s(G_1 \cdots G_{j-1}(F_j - G_j)F_{j+1} \cdots F_n) = \int \hat{f}_{1,k_1}(\alpha_1) \cdots \hat{f}_{j-1,k_{j-1}}(\alpha_{j-1}) \omega_s \left(\mathrm{e}^{\mathrm{i}\alpha_1 A_s^{(1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_{j-1} A_s^{(j-1)}} (F_j - G_j) F_{j+1} \ldots F_n \right) \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{j-1},$$

and invoking the Cauchy-Schwarz inequality for ω_s we can write, using Definition 4,

$$\begin{aligned} \left|\omega_s \left(\mathrm{e}^{\mathrm{i}\alpha_1 A_s^{(1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_{j-1} A_s^{(j-1)}} \ (F_j - G_j) \ F_{j+1} \ldots F_n \right)\right|^2 &\le \omega_s(F_n^* \cdots F_{j+1}^* F_{j+1} \cdots F_n)^{1/2} \omega_{js}^-(\alpha_1, \ldots, \alpha_{j-1}; (F_j - G_j)^2)^{1/2} \\ &\le R^{2(n-j)} \sigma_{js}^- \left(\alpha_1, \ldots, \alpha_{j-1}; \left|f_j - f_{j,k_j}\right|^2 \right), \end{aligned}$$

from which we obtain

$$|\omega_s(G_1 \cdots G_{j-1}(F_j - G_j)F_{j+1} \cdots F_n)| \le R^n \int |\hat{f}_{1,k_1}(\alpha_1)| \cdots |\hat{f}_{j-1,k_{j-1}}(\alpha_{j-1})| \, \sigma_{js}^- \left(\alpha_1, \ldots, \alpha_{j-1}; \left|f_j - f_{j,k_j}\right|^2 \right)^{1/2} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{j-1}. \tag{17}$$

Furthermore, Lemma 6 and the dominated convergence theorem allow us to conclude that

$$\limsup_{t\to\infty} |\omega_t(G_1 \cdots G_{j-1}(F_j - G_j)F_{j+1} \cdots F_n)| \le R^n \int |\hat{f}_{1,k_1}(\alpha_1)| \cdots |\hat{f}_{j-1,k_{j-1}}(\alpha_{j-1})| \, \sigma_{j\infty}^- \left(\alpha_1, \ldots, \alpha_{j-1}; \left|f_j - f_{j,k_j}\right|^2 \right)^{1/2} \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{j-1}. \tag{18}$$

The terms of the second sum on the right hand side of (16) can be handled in a similar way, leading to the estimates

$$|\omega_s(G_1 \cdots G_J F_{J+1} \cdots F_{j-1}(F_j - G_j)G_{j+1} \cdots G_n)| \le R^n \int |\hat{f}_{j+1,k_{j+1}}(\alpha_{j+1})| \cdots |\hat{f}_{n,k_n}(\alpha_n)| \, \sigma_{js}^+ \left(\alpha_{j+1}, \ldots, \alpha_n; \left|f_j - f_{j,k_j}\right|^2 \right)^{1/2} \mathrm{d}\alpha_{j+1} \cdots \mathrm{d}\alpha_n, \tag{19}$$

and

$$\limsup_{t\to\infty} |\omega_t(G_1 \cdots G_J F_{J+1} \cdots F_{j-1}(F_j - G_j)G_{j+1} \cdots G_n)| \le R^n \int |\hat{f}_{j+1,k_{j+1}}(\alpha_{j+1})| \cdots |\hat{f}_{n,k_n}(\alpha_n)| \, \sigma_{j\infty}^+ \left(\alpha_{j+1}, \ldots, \alpha_n; \left|f_j - f_{j,k_j}\right|^2 \right)^{1/2} \mathrm{d}\alpha_{j+1} \cdots \mathrm{d}\alpha_n. \tag{20}$$

Combining estimates (17), (18), (19) and (20) with identity (16) leads to

$$\limsup_{t\to\infty} |\Delta_t| \le 2R^n D(k_1,\ldots,k_n), \tag{21}$$

for any $k_1,\ldots,k_n\in\mathbb{N}$, where

$$D(k_1,\ldots,k_n) = \sum_{j=1}^{J}\int |\hat f_{1,k_1}(\alpha_1)|\cdots|\hat f_{j-1,k_{j-1}}(\alpha_{j-1})|\,\sigma_{j\infty}^-\left(\alpha_1,\ldots,\alpha_{j-1};\left|f_j-f_{j,k_j}\right|^2\right)^{1/2}\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_{j-1}$$
$$+\sum_{j=J+1}^{n}\int |\hat f_{j+1,k_{j+1}}(\alpha_{j+1})|\cdots|\hat f_{n,k_n}(\alpha_n)|\,\sigma_{j\infty}^+\left(\alpha_{j+1},\ldots,\alpha_n;\left|f_j-f_{j,k_j}\right|^2\right)^{1/2}\mathrm{d}\alpha_{j+1}\cdots\mathrm{d}\alpha_n.$$

By Lemma 9 and our assumptions, $\lim_k |f_j - f_{j,k}| = 0$ holds $\sigma_{j\infty}^-(\alpha_1,\ldots,\alpha_{j-1};\,\cdot\,)$ almost everywhere for all $j\in\{1,\ldots,J\}$ and $\alpha_1,\ldots,\alpha_{j-1}\in\mathbb{R}$. It follows from the dominated convergence theorem that

$$\lim_k \sigma_{j\infty}^-\left(\alpha_1,\ldots,\alpha_{j-1};|f_j-f_{j,k}|^2\right)=0.$$

In a similar way one shows that

$$\lim_k \sigma_{j\infty}^+\left(\alpha_{j+1},\ldots,\alpha_n;|f_j-f_{j,k}|^2\right)=0,$$

for all $j\in\{J+1,\ldots,n\}$ and $\alpha_{j+1},\ldots,\alpha_n\in\mathbb{R}$. Applying once again the dominated convergence theorem one concludes that

$$\lim_k\int |\hat f_{1,k_1}(\alpha_1)|\cdots|\hat f_{j-1,k_{j-1}}(\alpha_{j-1})|\,\sigma_{j\infty}^-\left(\alpha_1,\ldots,\alpha_{j-1};|f_j-f_{j,k}|^2\right)^{1/2}\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_{j-1}=0, \tag{22}$$

for all $j\in\{1,\ldots,J\}$ and $k_1,\ldots,k_{j-1}\in\mathbb{N}$, while

$$\lim_k\int |\hat f_{j+1,k_{j+1}}(\alpha_{j+1})|\cdots|\hat f_{n,k_n}(\alpha_n)|\,\sigma_{j\infty}^+\left(\alpha_{j+1},\ldots,\alpha_n;|f_j-f_{j,k}|^2\right)^{1/2}\mathrm{d}\alpha_{j+1}\cdots\mathrm{d}\alpha_n=0, \tag{23}$$

for all $j\in\{J+1,\ldots,n\}$ and $k_{J+1},\ldots,k_n\in\mathbb{N}$. The result now follows from (21) and the fact that

$$\lim_{k_n}\lim_{k_{n-1}}\cdots\lim_{k_{J+1}}\lim_{k_1}\lim_{k_2}\cdots\lim_{k_J} D(k_1,\ldots,k_n)=0,$$

a direct consequence of (22) and (23). □

3.3 Proof of Lemma 6

By Definition 4 we have

$$\sigma_{jt}^-(\alpha_1,\ldots,\alpha_{j-1};g)=\omega_{jt}^-(\alpha_1,\ldots,\alpha_{j-1};g(A_t^{(j)})),$$

for $g\in\mathcal{B}$ and Assumption (A) translates into

$$\lim_{t\to\infty}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\alpha x}\,\sigma_{jt}^-(\alpha_1,\ldots,\alpha_{j-1};\mathrm{d}x)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\alpha x}\,\sigma_{j\infty}^-(\alpha_1,\ldots,\alpha_{j-1};\mathrm{d}x).$$

The classical Lévy-Cramér continuity theorem readily implies that

$$\lim_{t\to\infty}\int_{\mathbb{R}} g(x)\,\sigma_{jt}^-(\alpha_1,\ldots,\alpha_{j-1};\mathrm{d}x)=\int_{\mathbb{R}} g(x)\,\sigma_{j\infty}^-(\alpha_1,\ldots,\alpha_{j-1};\mathrm{d}x),$$

for any $g\in\mathcal{B}$ such that $\sigma_{j\infty}^-(\alpha_1,\ldots,\alpha_{j-1};\mathcal{D}(g))=0$. This proves (11). Completely similar arguments prove (13) and (15). □

3.4 Proof of Lemma 7

Let $j \in \{1, \ldots, n\}$, $\sigma \in S_j^- \cup S_j^+$ and $E \subset \mathbb{R}$ a Borel set. Denoting $P = \mathbb{1}_E(A_\infty^{(j)})$, there exists a unitary $U \in \mathfrak{M}_\infty$ such that $\sigma(E) = \omega_\infty(U^*PU)$. Since both operators P and U^*PU are positive, the faithfulness of ω_∞ implies that

$$\begin{aligned}\sigma(E) = 0 \quad &\Leftrightarrow \quad U^*PU = 0\\ &\Leftrightarrow \quad P = 0\\ &\Leftrightarrow \quad \omega_\infty(P) = 0\\ &\Leftrightarrow \quad \omega_\infty^{(j)}(E) = 0,\end{aligned}$$

and so σ and $\omega_\infty^{(j)}$ are equivalent measures. □

4 Applications and discussion

4.1 A simple application

We first recall some standard results of non-commutative probability theory refering the reader to [Bia] or [Mey] for proofs. We consider an orthonormal basis $\{\Omega, X\}$ of the Hilbert space $\mathbb{C}^2$, and the basis $a^\times, a^+, a^-, a^\circ$ of the algebra $M(2, \mathbb{C})$ of complex, 2×2 matrices defined by

$$a^\times \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a^+ \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^- \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^\circ \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

in the basis $\{\Omega, X\}$.

For any $m \in \mathbb{N}^*$ let $\mathrm{T}\Phi_m \equiv (\mathbb{C}^2)^{\otimes m}$, the m-fold tensor product of $\mathbb{C}^2$. For $i \in \{1, \ldots, m\}$ and $\varepsilon \in \{\times, +, -, \circ\}$ we denote by a_i^ε the ampliation of a^ε acting on the i-th copy of $\mathbb{C}^2$ in $\mathrm{T}\Phi_m$. The family $(a_i^\varepsilon)_{i \in \{1,\ldots,m\}, \varepsilon \in \{\times,+,-,\circ\}}$ is then a basis of the algebra $\mathfrak{M}_m \equiv M(2, \mathbb{C})^{\otimes m}$ (this is the toy Fock space approximation of [A]). We further denote by Ω_m the vector $\Omega^{\otimes m}$ and by ω_m the associated state $A \mapsto (\Omega_m, A\Omega_m)$. In this state, the operators

$$n_{i,m} \equiv \frac{a_i^+ + a_i^-}{\sqrt{m}}, \qquad p_{i,m} \equiv \frac{a_i^+ + a_i^-}{\sqrt{m}} + a_i^\circ + \frac{1}{m} a_i^\times, \qquad z_{i,m} \equiv a_i^\circ,$$

respectively follow the laws

$$\frac{1}{2}\,\delta_{-m^{-1/2}} + \frac{1}{2}\,\delta_{m^{-1/2}}, \qquad \frac{m}{m+1}\,\delta_0 + \frac{1}{m+1}\,\delta_{1+m^{-1}}, \qquad \delta_0,$$

with characteristic functions

$$\cos\left(\frac{\alpha}{\sqrt{m}}\right), \qquad \frac{\mathrm{e}^{\mathrm{i}\alpha(1+1/m)} + m}{1+m}, \qquad 1.$$

Moreover, for $b \in \{n, p, z\}$, $b_{i,m}$ and $b_{j,m}$ commute if $i \neq j$. Therefore each family $(b_{i,m})_{i \in \{1,\ldots,m\}}$, has a joint law in the state ω_m and, in addition, this joint law can be seen to correspond to independent random variables. It follows that the random variables

$$N_n \equiv \sum_{i=1}^{m} n_{i,m}, \qquad P_m \equiv \sum_{i=1}^{m} p_{i,m}, \qquad Z_m \equiv \sum_{i=1}^{m} z_{i,m},$$

have characteristic functions

$$\left[\cos\left(\frac{\alpha}{\sqrt{m}}\right)\right]^m, \qquad \left[\frac{\mathrm{e}^{\mathrm{i}\alpha(1+1/m)} + m}{1+m}\right]^m, \qquad 1,$$

which, as $m \to \infty$, are easily seen to converge to

$$\mathrm{e}^{-\alpha^2/2}, \qquad \mathrm{e}^{\mathrm{e}^{\mathrm{i}\alpha}-1}, \qquad 1,$$

the characteristic functions of the centered normal law with variance 1, the Poisson law of intensity 1 and the law δ_0. The classical Lévy-Cramér theorem thus implies the convergence in law of the individual sequences

$(N_m)_{m\in\mathbb{N}^*}$, $(P_m)_{m\in\mathbb{N}^*}$, $(Z_m)_{m\in\mathbb{N}^*}$ to the corresponding random variables. As an application of our quantum Lévy-Cramér theorem, we shall now consider some properties of the joint sequence $(N_m, P_m, Z_m)_{m\in\mathbb{N}^*}$.

We start with a simple observation. Denote by $\Phi_m \subset \mathrm{T}\Phi_m$ the subspace generated by completely symmetric tensor products. An orthonormal basis of Φ_m is given by the family $(e_k)_{k\in\{0,\ldots,m\}}$ where e_k is the (normalized) complete symmetrization of $X^{\otimes k}\otimes\Omega^{\otimes(m-k)}$. In particular $\Omega_m = e_0$. The operators N_m, P_m, Z_m clearly leave Φ_m invariant. A simple calculation shows that

$$
\begin{aligned}
N_m e_k &= \sqrt{1-\frac{k-1}{m}}\,\sqrt{k}\,e_{k-1} + \sqrt{1-\frac{k}{m}}\,\sqrt{k+1}\,e_{k+1},\\
P_m e_k &= \sqrt{1-\frac{k-1}{m}}\,\sqrt{k}\,e_{k-1} + \sqrt{1-\frac{k}{m}}\,\sqrt{k+1}\,e_{k+1} + \left(k+1-\frac{k}{m}\right)\,e_k,\\
Z_m e_k &= k\,e_k,
\end{aligned}
\tag{24}
$$

where, by convention, $e_{-1} = e_{m+1} = 0$. In studying the random variables N_m, P_m, Z_m in the state ω_m we may therefore consider that these operators act on the space Φ_m.

To describe limiting random variables $N_\infty, P_\infty, Z_\infty$ we denote by $\mathfrak{M}_\infty$ the algebra of bounded operators on $\Phi \equiv \ell^2(\mathbb{N})$ with $(\tilde e_k)_{k\in\mathbb{N}}$ its canonical orthonormal basis, and by ω_∞ the state $A \mapsto (\tilde e_0, A\tilde e_0)$. The operators a^+, a^-, a° defined by

$$
a^+\tilde e_k \equiv \sqrt{k+1}\,\tilde e_{k+1}, \qquad a^-\tilde e_k \equiv \sqrt{k}\,\tilde e_{k-1}, \qquad a^\circ\tilde e_k \equiv a^+a^-\tilde e_k = k\,\tilde e_k,
$$

(with the convention $\tilde e_{-1} = 0$) are such that in the state ω_∞, for any $w\in\mathbb{C}$, the operators

$$
wa^+ + \bar w a^-, \qquad wa^+ + \bar w a^- + a^\circ + |w|^2 I, \qquad a^\circ,
\tag{25}
$$

follow respectively a centered normal law with variance $|w|^2$, a Poisson law of intensity $|w|^2$ and the law δ_0. Setting

$$
N_\infty \equiv a^+ + a^-, \qquad P_\infty \equiv a^+ + a^- + a^\circ + I, \qquad Z_\infty \equiv a^\circ,
$$

we therefore have the convergences in law $N_m \to N_\infty$, $P_m \to P_\infty$ and $Z_m \to Z_\infty$.

Let us show that Assumption (A) holds with $(A_m^{(1)}, A_m^{(2)}, A_m^{(3)}) \equiv (N_m, P_m, Z_m)$, $m\in\mathbb{N}^*\cup\{\infty\}$. We first note that the partial isometry $S_m : \Phi_m \to \Phi$ induced by the map $e_k \mapsto \tilde e_k$, $k\in\{0,\ldots,m\}$ satisfies

$$
\omega_m(\mathrm{e}^{\mathrm{i}\alpha_1 A_m^{(j_1)}}\cdots\mathrm{e}^{\mathrm{i}\alpha_k A_m^{(j_k)}}) = \omega_\infty(\mathrm{e}^{\mathrm{i}\alpha_1 \tilde A_m^{(j_1)}}\cdots\mathrm{e}^{\mathrm{i}\alpha_k \tilde A_m^{(j_k)}}),
\tag{26}
$$

where $\tilde A_m^{(j)} \equiv S_m A_m^{(j)} S_m^*$. Using relations (24), one easily shows that, for any $k\in\mathbb{N}$ and $j\in\{1,2,3\}$

$$
\lim_{m\to\infty} \tilde A_m^{(j)}\tilde e_k = A_\infty^{(j)}\tilde e_k.
\tag{27}
$$

Using the fact that the set of finite linear combinations of basis vectors $\tilde e_k$ is a common core for all $\tilde A_m^{(j)}$ and $A_\infty^{(j)}$, it follows from (27) that the sequence $\tilde A_m^{(j)}$ converges to $A_\infty^{(j)}$ in strong resolvent sense (see *e.g.*, Theorem VIII.25 in [RS]). On concludes that

$$
\mathrm{s}{-}\lim_{m\to\infty} \mathrm{e}^{\mathrm{i}\alpha\tilde A_m^{(j)}} = \mathrm{e}^{\mathrm{i}\alpha A_\infty^{(j)}},
$$

for any $\alpha\in\mathbb{R}$ and $j\in\{1,2,3\}$. Assumption (A) clearly follows from this relation and Equ. (26).

Computations using commutation between Weyl operators (see [Bia]) show that

$$
\mathrm{e}^{\mathrm{i}\alpha N_\infty}\,\mathrm{e}^{\mathrm{i}\beta P_\infty}\,\mathrm{e}^{-\mathrm{i}\alpha N_\infty} = \mathrm{e}^{\mathrm{i}\beta((1-\mathrm{i}\alpha)a^+ + (1+\mathrm{i}\alpha)a^- + a^\circ + |1-\mathrm{i}\alpha|^2)},
$$

so that (recall (25))

$$
\beta \mapsto \omega_\infty\left(\mathrm{e}^{\mathrm{i}\alpha N_\infty}\,\mathrm{e}^{\mathrm{i}\beta P_\infty}\,\mathrm{e}^{-\mathrm{i}\alpha N_\infty}\right),
$$

is the characteristic function of a Poisson law of intensity $|1-\mathrm{i}\alpha|^2$. We therefore obtain a non-trivial consequence of Theorem 2:

$$
\lim_{m\to\infty} \omega_m\big(f_1(N_m)f_2(P_m)f_3(N_m)\big) = \omega_\infty\big(f_1(N_\infty)f_2(P_\infty)f_3(N_\infty)\big)
$$

for any $f_1, f_2, f_3 \in \mathcal{B}$ with f_2 continuous at every point of $\mathbb{N}$. In particular, for any $a<b$ in $\mathbb{R}$ and any $c<d$ in $\mathbb{R}\setminus\mathbb{N}$,

$$
\lim_{m\to\infty} \omega_m\big(\mathbb{1}_{[a,b]}(N_m)\mathbb{1}_{[c,d]}(P_m)\mathbb{1}_{[a,b]}(N_m)\big) = \omega_\infty\big(\mathbb{1}_{[a,b]}(N_\infty)\mathbb{1}_{[c,d]}(P_\infty)\mathbb{1}_{[a,b]}(N_\infty)\big).
$$

In a similar way, one shows that

$$\lim_{m\to\infty} \omega_m\big(f_1(P_m)f_2(N_m)f_3(P_m)\big) = \omega_\infty\big(f_1(P_\infty)f_2(N_\infty)f_3(P_\infty)\big)$$

for any $f_1, f_2, f_3 \in \mathcal{B}$.

This example also allows us to illustrate the necessity of our strengthened continuity assumptions (note that the state ω_∞ is not faithful, as for example Z_∞ is a positive operator and yet $\omega_\infty(Z_\infty) = 0$). For finite m it follows from (24) that Z_m is a positive matrix with integer eigenvalues, so that $\mathbb{1}_{\{1\}}(Z_m + \frac{1}{m}) = 0$ and hence

$$\omega_m\left(\mathrm{e}^{\mathrm{i}\alpha N_m}\mathbb{1}_{\{1\}}\left(Z_m + \frac{1}{m}\right)\mathrm{e}^{-\mathrm{i}\alpha N_m}\right) = 0.$$

On the other hand,

$$\omega_\infty\big(\mathrm{e}^{\mathrm{i}\alpha N_\infty}\mathbb{1}_{\{1\}}(Z_\infty)\mathrm{e}^{-\mathrm{i}\alpha N_\infty}\big) = \omega_\infty\big(\mathbb{1}_{\{1\}}(\mathrm{e}^{\mathrm{i}\alpha N_\infty}Z_\infty\mathrm{e}^{-\mathrm{i}\alpha N_\infty})\big), \tag{28}$$

and $\mathrm{e}^{\mathrm{i}\alpha N_\infty}Z_\infty\mathrm{e}^{-\mathrm{i}\alpha N_\infty} = -\mathrm{i}\alpha a^+ + \mathrm{i}\alpha a^- + a^\circ + |\alpha|^2$ (again by the commutation relations of Weyl operators, see [Bia]). Since this operator follows, in the state ω_∞, a Poisson law of intensity α^2, the right hand side of Equ. (28) is strictly positive provided $\alpha \neq 0$. We thus have

$$\omega_\infty\big(\mathrm{e}^{\mathrm{i}\alpha N_\infty}\mathbb{1}_{\{1\}}(Z_\infty)\mathrm{e}^{-\mathrm{i}\alpha N_\infty}\big) \neq \lim_{m\to\infty}\omega_m\left(\mathrm{e}^{\mathrm{i}\alpha N_m}\mathbb{1}_{\{1\}}\left(Z_m + \frac{1}{m}\right)\mathrm{e}^{-\mathrm{i}\alpha N_m}\right),$$

even though assumption (A) obviously remains true if we replace Z_m by $Z_m + \frac{1}{m}$. This shows that Theorem 2 is false if we only assume that each f_j satisfies $\omega_\infty^{(j)}(\mathcal{D}(f_j)) = 0$, and illustrates why: the projection associated with the eigenvalue 1 of Z_∞ is not in the support of ω_∞, and so the singularities of $\mathbb{1}_{\{1\}}$ have measure zero under the law of Z_∞. However, when conjugated by $\mathrm{e}^{\mathrm{i}\alpha N_\infty}$, this projection is sent to the support of ω_∞ and the singularities of $\mathbb{1}_{\{1\}}$ have non-zero measure under the law of $\mathrm{e}^{\mathrm{i}\alpha N_\infty}Z_\infty\mathrm{e}^{-\mathrm{i}\alpha N_\infty}$.

4.2 Previous results of Lévy-Cramér type

The paper [CH] was the first to study explicitly a non-commutative central limit theorem, which it proves using a result of the non-commutative Lévy-Cramér type (Theorem 2 in the cited paper). That result, in a slightly simplified framework, is the following: consider a sequence $(q_n, p_n)_{n\in\mathbb{N}}$ of canonical pairs on $\mathcal{H} \equiv \mathrm{L}^2(\mathbb{R})$, *i.e.* a pair of (unbounded) self-adjoint operators such that there exists a dense subspace $\mathcal{D}_n \subset \mathcal{H}$ in the domain of both q_n and p_n, which is stable by q_n and p_n, on which the canonical commutation relation (CCR)

$$q_n p_n - p_n q_n = \mathrm{i}I, \tag{29}$$

holds. In analytically simpler terms, this can be rewritten as the Weyl relation

$$\mathrm{e}^{\mathrm{i}(xp_n + yq_n)} = \mathrm{e}^{\mathrm{i}xp_n}\mathrm{e}^{\mathrm{i}yq_n}\mathrm{e}^{\mathrm{i}xy/2}, \tag{30}$$

(see [BR1] or [Pet] for more details on canonical pairs).

Assume that every (q_n, p_n) is irreducible, *i.e.* no nontrivial subspace of $\mathcal{H}$ is left invariant by all operators $\mathrm{e}^{\mathrm{i}(xp_n+yq_n)}$. A normal reference state ρ on $\mathcal{B}(\mathcal{H})$ is fixed; then a state ρ_n on $\mathcal{B}(\mathcal{H})$ can be associated to every canonical pair (q_n, p_n) by

$$\rho_n(A) \equiv \rho(U_n^{-1}AU_n),$$

where U_n is the unitary operator mapping (q_n, p_n) to the Schrödinger representation of the CCR (29). The Stone-von Neumann unicity theorem for irreducible representations of the CCR ensures its existence (see [Mey]).

Cushen and Hudson define the pseudo-characteristic function by

$$\varphi_n(x, y) \equiv \rho(\mathrm{e}^{\mathrm{i}(xp_n+yq_n)}).$$

This definition is different from ours but in the case of canonical pairs it follows from (30) that the two definitions are essentially equivalent. It is then proven that there exists a state ρ_∞ such that

$$\lim_{n\to\infty}\rho_n(A) = \rho_\infty(A), \tag{31}$$

for every $A \in \mathcal{B}(\mathcal{H})$ (a property which Cushen and Hudson call convergence in distribution) if and only if the sequence φ_n converges pointwise on $\mathbb{R}^2$ to a function which is continuous at zero.

It is the easy part of the theorem to show that, if the sequence $(\varphi_n)_{n\in\mathbb{N}}$ has a pointwise limit φ_∞ which is continuous at zero, then φ_∞ is of the form

$$\varphi_\infty(x,y)=\rho(\mathrm{e}^{\mathrm{i}(xp_\infty+yq_\infty)}),$$

for some canonical pair (q_∞,p_∞). This and the Weyl relation (30) imply that pointwise convergence of φ_n to a function which is continuous at zero is equivalent to our assumption (A) for $\omega=\rho$ and $A_n^{(1)}=q_n$, $A_n^{(2)}=p_n$. Moreover, the Weyl relation (30) implies that

$$\mathrm{e}^{\mathrm{i}xp_n}\mathrm{e}^{\mathrm{i}yq_n}\mathrm{e}^{-\mathrm{i}xp_n}=\mathrm{e}^{-ixy}\mathrm{e}^{\mathrm{i}yq_n},\qquad \mathrm{e}^{\mathrm{i}yq_n}\mathrm{e}^{\mathrm{i}xp_n}\mathrm{e}^{-\mathrm{i}yq_n}=\mathrm{e}^{+ixy}\mathrm{e}^{\mathrm{i}xp_n},$$

and the law of both p_∞ and q_∞ in the state ω_∞ is Gaussian, so that Theorem 2 implies (7) for all bounded Borel functions. The conclusion of [CH], *i.e.* the convergence (31), is stronger than ours at first sight, but it is a consequence of the properties of the Weyl correspondence (as described in the proofs of Proposition 6 and Theorem 2 of that paper) that both conclusions are actually equivalent. Our results therefore extend the results of Cushen and Hudson, which rely heavily on the particular properties of canonical pairs.

The other occurrence of a non-commutative Lévy-Cramér type result we are aware of is [Kup]. In this paper, Kuperberg proves implications of pointwise convergence of pseudo-characteristic functions, of the same type as (7): for $\mathfrak{M}$ a von Neumann algebra equipped with a normal state ρ, he considers for any $A\in\mathfrak{M}$ the elements

$$A_N=\frac{1}{\sqrt{N}}\sum_{k=1}^{N}I^{\otimes k-1}\otimes A\otimes I^{\otimes N-k},$$

(we use different notations from that in [Kup] to stay as close as possible to our own) and shows that, for fixed self adjoint elements $A^{(1)},\ldots,A^{(k)}\in\mathfrak{M}$, for any self adjoint non commutative polynomial p, $p(A_N^{(1)},\ldots,A_N^{(k)})$ converges in distribution to $p(X^{(1)},\ldots,X^{(k)})$ in any *tracial* state ρ, where $(X^{(1)},\ldots,X^{(k)})$ is a (classical) centered Gaussian vector with covariance matrix $C_{ij}\equiv\rho(A^{(i)}A^{(j)})-\rho(A^{(i)})\rho(A^{(j)})$. One of the steps of that proof is to show that the convergence of $\rho\big(\mathrm{e}^{\mathrm{i}\alpha_1A^{(1)}}\cdots\mathrm{e}^{\mathrm{i}\alpha_kA^{(k)}}\big)$ to $\mathbb{E}\big(\mathrm{e}^{\mathrm{i}\alpha_1X^{(1)}}\cdots\mathrm{e}^{\mathrm{i}\alpha_kX^{(k)}}\big)$ for every $\alpha_1,\ldots,\alpha_k$ implies the convergence of any quantity $\rho\big(f(A^{(1)})\cdots f(A^{(k)})\big)$ to $\mathbb{E}\big(f(X^{(1)})\cdots f(X^{(k)})\big)$ for every $f_1,\ldots,f_k\in\mathcal{C}$. The same method could easily be extended to include bounded Borel functions with the standard continuity assumptions (here the limiting quantities are purely commutative) but the proof here uses the fact that the GNS norm associated with the reference state ρ is spectral, which is only true if ρ is tracial. Our result therefore improves the scope of application of this part of Kuperberg's results.

4.3 Applications

In this subsection we discuss the literature related to Assumption (A), *i.e.* the specific models for which (A) has been verified and to which our main results apply. The related results cover a number of technically and conceptually different frameworks and, for reason of space, we shall only briefly touch on several central points. For the terminology and additional information we refer the reader to the original papers.

The first series of results originates in the paper [AFL] (later extended in more than one direction, see *e.g.* [Gou] and references therein) for the weak coupling limit and [AL] for the low density limit. The results of the form of (A) in the cited papers exist at two different levels. First there are kinematical results: Theorem 3.4 in [AFL] and Lemma 2.1 in [AL]. Note that, at this level, the von Neumann algebras $\mathfrak{M}_t$ and the states ω_t are the same (the algebra being of the form $\mathcal{B}(\mathcal{H})$, the state being the pure state associated with the vacuum vector) for all t in the case of the weak coupling limit, but not in the case of the low density limit, where the parameter z enters the definition of the considered scalar product. The second level at which these papers prove results of the form (A) is the dynamical one: Theorem (II) in [AFL], Theorem 5.1 in [AL] where this time the structure depends on t in all cases. These theorems consider only one unitary U_t at a time; we can, however, make a connection with a non-trivial form of (A) by noting that they can be easily extended to the case where the single unitary operator U_t is replaced with a product of different operators corresponding to different couplings V in the Hamiltonian.

Another possible application comes from [AP] (and its extension in [AJ]); this time the considered limit is that of "repeated to continuous" interactions. The whole picture, that is both the $h>0$ systems and the limiting case can be described within a single algebra. Here again the main result of the paper, Theorem 13, shows *a priori* a result of type (A) for the case of a single operator, but Corollary 18 implies that one has, in the common Hilbert space for all operators, strong convergence of the operators $\mathrm{e}^{\mathrm{i}\alpha_iA_n^{(i)}}$ to $\mathrm{e}^{\mathrm{i}\alpha_iA_\infty^{(i)}}$. Therefore, (A) will also hold for a product of more that one operator, corresponding to possibly different operators L (when using Theorem 17) or different Hamiltonians H (when using Theorem 19) – it is Theorem 19 that we used in subsection 4.1, in the simple case where $\mathcal{H}_0 = \mathbb{C}$. Note that standard results on strong resolvent convergence of operators (see [RS]) imply the strong convergence of $f_i(A_n^{(i)})$ to $f_i(A_\infty^{(i)})$, hence the convergence (7) for bounded *continous* functions f_i. It is a non-trivial improvement to obtain the same result for non-continuous f_i.

Finally, we mention the study of "fluctuation algebras" in the papers [GV], [GVV] (and subsequent papers by the same authors), [Mat], [AJPP] and [JPP]. These papers consider operators $A_n^{(i)}$ or $A_t^{(i)}$ of the form

$$A_n^{(i)} = \frac{1}{\sqrt{n}} \sum_{|x|\leq n} (\tau_x(A_i) - \omega(A_i)),$$

or

$$A_t^{(i)} = \frac{1}{\sqrt{t}} \int_0^t (\tau^s(A_i) - \omega(A_i))\,\mathrm{d}s,$$

where τ_x is a translation operator (as in [GV], [GVV], [Mat], which study *spatial* fluctuations) or τ^s is a dynamical group (as in [AJPP] and [JPP], which study *time* fluctuations) and every A_i is an observable of the considered system (belonging to some subalgebra $\mathfrak{M}_1$). In both cases, a result of the type (A) is proven in which the limiting quantities $\omega_\infty(\mathrm{e}^{\mathrm{i}\alpha_1 A_\infty^{(1)}} \cdots \mathrm{e}^{\mathrm{i}\alpha_p A_\infty^{(p)}})$ are of the form $\rho(W(A^{(1)}) \ldots W(A^{(p)}))$, where the W are elements of a Weyl algebra over $\mathfrak{M}_1$ for an explicit symplectic form, and ρ is a quasi-free state on this Weyl algebra (see [BR1] or [Pet]). A case of particular interest is when this symplectic form is found to be null, so that the Weyl algebra is abelian; in this case, the law of the operators $A_\infty^{(1)}, \ldots, A_\infty^{(n)}$ in the state ω_∞ is that of a (classical) Gaussian vector.

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