# Scattering from Sparse Potentials: a deterministic approach

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**Abstract.** Completeness of the wave operators has been proven for a family of random Schrödinger operators with sparse potentials in the recent paper [17], using a probabilistic approach. As mentioned at Voss, a deterministic result in this direction can also be derived from a Jakšić–Last criterion of completeness [7] and Fredholm's theorem. We present this approach.

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## 1. Introduction

Since their introduction by Anderson [1], there has been considerable interest in Schrödinger operators with random potentials. In the classical case, these operators represent the energy of a particle affected by a random potential supported on a lattice. They are of the form $H = \Delta + \lambda V$, where $\Delta$ is the centered, discrete Laplacian on $\mathbb{Z}^d$, $\lambda$ is a real parameter (the so-called *disorder*) and $V$ is a random potential on $\mathbb{Z}^d$. In [1], Anderson anticipated the spectral structure of $H$ (i.e., the intervals of localization/delocalization) with respect to the disorder. While the localization aspect of the Anderson conjecture has been mathematically settled in the seminal papers [5, 4, 2, 3], practically nothing is known about the delocalization aspect.

Several research teams have also studied various *sparse models* [6, 9, 10, 11, 12, 13, 14, 15, 17]. In these nonergodic models, spectral properties of $H$ are expected to follow from various geometric constraints on the sites of the potential, having in common that the minimal distance between two sites becomes arbitrarily large when removing a finite number of them. Examples have been exhibited where all the expected spectral properties are satisfied (almost surely), including completeness [17] of the wave operators on the spectrum of $\Delta$. We present a deterministic extension of this last result.

We consider a discrete Schrödinger operator $H = \Delta + V$ in dimension $d \geqslant 2$, where $\Delta$ is the centered Laplacian and $V$ is a bounded potential: for $\varphi \in l^2(\mathbb{Z}^d)$ and $n \in \mathbb{Z}^d$

$$(H\varphi)(n) = \sum_{|m-n|_1=1} \varphi(m) + V(n),$$

where $|n|_1 = \sum_{j=1}^d |n^{(j)}|$. We assume that the support of $V$, which we denote by $\Gamma \subset \mathbb{Z}^d$, satisfies the following sparseness assumption:

**(A)** *There exists an* $\epsilon > 0$ *such that* $\sum_{m\in\Gamma\setminus\{n\}} |n-m|^{-\frac{1}{2}+\epsilon}$ *is finite for all* $n \in \Gamma$ *and tends to* 0 *when* $|n| \to \infty$ *in* $\Gamma$.

This is the case, for instance, if $\Gamma = \{(j^4, 0, \ldots, 0) \in \mathbb{Z}^d \ ; \ j \in \mathbb{Z}\}$.

Recall [18] that the *wave operators* on a Borel set $\Theta \subset \mathbb{R}$ are the strong limits $\Omega^\pm(H,\Delta) = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}t\Delta}\mathbf{1}_\Theta(\Delta)$ (if they exist); they are *complete* if their range is $\mathrm{Ran}\mathbf{1}_\Theta(H)$ (so $\Delta$ and $H$ are unitarily equivalent on $\Theta$). We prove:

**Theorem.** *Assume* (A)*. Then, the wave operators* $\Omega^\pm(H,\Delta)$ *exist on* $[-2d, 2d]$*. Moreover, they are complete on* $[-2d, 2d]$ *minus a set of Lebesgue measure zero.*

It is possible to remove the exceptional set in the above by working in the random frame. Then $\{V(n)\}_{n\in\Gamma}$ is a family of independent, identically distributed, absolutely continuous random variables whose common density is compactly supported.[1] Since the essential support of the absolutely continuous spectrum of $\Delta$ is $[-2d, 2d]$, and since under Assumption (A) the wave operators exist on this last interval, the Jakšić–Last theorem [8] and the above immediately yield:

**Corollary.** *In the random frame, Assumption* (A) *implies that the wave operators exist and are complete on* $[-2d, 2d]$*, almost surely.*

This conclusion is stronger than the one we obtained in [17], where only completeness of the wave operators is derived. However, our present assumption is also stronger, since unbounded potentials are discarded.

Here is the outline of the paper. In the sequel $\{n \in \mathbb{Z}^d \ ; \ \inf_{m\in\Gamma} |n-m|_1 \leqslant 1\}$ is denoted by $\Gamma_1$, while $\mathbf{1}_0$ and $\mathbf{1}_1$ denote the projections onto $l^2(\Gamma)$ and $l^2(\Gamma_1)$ respectively. Moreover $\delta_m(n)$ denotes the Kronecker delta, where $m, n \in \mathbb{Z}^d$. For $z \in \mathbb{C}_+$ we consider the following restrictions of the free and perturbed resolvents,

$$F_1(z) = \mathbf{1}_1(\Delta - z)^{-1}\mathbf{1}_1, \quad F_0(z) = \mathbf{1}_0(\Delta - z)^{-1}\mathbf{1}_0,$$

$$P_1(z) = \mathbf{1}_1(H - z)^{-1}\mathbf{1}_1, \quad P_0(z) = \mathbf{1}_0(H - z)^{-1}\mathbf{1}_0.$$

Our study is based the following theorem of Jakšić and Last [7]:

[1] Explicitly, the probability space is given by $\mathbb{R}^\Gamma$ equipped with its Borel $\sigma$-algebra and a probability measure $\prod_\Gamma \mu$, where $\mu$ is an absolutely continuous, compactly supported measure on $\mathbb{R}$. The variable $V(n)$ is then the projection on the $n^{th}$ coordinate, for $n \in \Gamma$.

**Proposition 1.** *Let $a < b$. Suppose $\|F_1(e + \mathrm{i}0)\| < \infty$ and $\|P_1(e + \mathrm{i}0)\| < \infty$ for all $e \in [a, b]$. Then, the wave operators exist and are complete on $[a, b]$.*

For $[a, b] \subset [-2d, 2d] \setminus (\{2d, 2d-4, \ldots, -2d+4, -2d\} \cup \{0\})$ and $z$ in the strip $\mathcal{S} := \{e + \mathrm{i}y \ ; \ a < e < b,\ 0 < y < 1\}$, the following *a priori* estimate [16] is available:

**Proposition 2.** *Let $n = |n|\omega \in \mathbb{Z}^d$. Then, $\lim_{z \to e, z \in \mathbb{C}_+} \langle \delta_0 \,|\, (\Delta - z)^{-1} \delta_n \rangle$ exists and is $O(|n|^{-\frac{1}{2}})$ uniformly in $(e, \omega) \in [a, b] \times \mathrm{S}^{d-1}$. More generally, $\langle \delta_0 \,|\, (\Delta - z)^{-1} \delta_n \rangle = O(|n|^{-\frac{1}{2}} \log |n|)$ uniformly in $(z, \omega) \in \overline{\mathcal{S}} \times \mathrm{S}^{d-1}$.*

Therefore, sparseness will ensure that the matrix elements of $F_1(z)$ are small except about its diagonal. By subtracting a block-diagonal to $F_1(z)$ the remaining part will be compact. We will derive the finiteness of $\|F_1(e + \mathrm{i}0)\|$ first, and then deduce the same for $\|P_1(e + \mathrm{i}0)\|$ by means of Fredholm's theorem:

**Proposition 3.** *Let $K(z)$ be a function with values in the space of compact operators (endowed with the uniform topology). Suppose $K(z)$ is continuous on $\overline{\mathcal{S}}$ and analytic in $\mathcal{S}$. Then, either $1 - K(z)$ is never invertible on $\overline{\mathcal{S}}$, or it is invertible except on a closed set of Lebesgue measure zero whose intersection with $\mathbb{C}_+$ consists of isolated points.*

## 2. Proof of the theorem

Let us partition $\Gamma_1$ as follows: for all $n \in \Gamma$, we select a neighborhood $\mathcal{B}(n) \subseteq \{m \in \Gamma_1 \ ; \ |m - n|_1 \leqslant 1\}$ containing $n$ in such a way that $\bigcup_{n \in \Gamma} \mathcal{B}(n) = \Gamma_1$ and $\mathcal{B}(n) \cap \mathcal{B}(n') = \emptyset$ if $n \neq n'$. For all $m \in \Gamma_1$ there exists exactly one $n \in \Gamma$ such that $m \in \mathcal{B}(n)$; we then set $\mathcal{B}(m) = \mathcal{B}(n)$.

For $n \in \Gamma_1$, let $S(n) = \sum_{m \in \Gamma_1 \setminus \mathcal{B}(n)} \sup_{z \in \mathcal{S}} |\langle \delta_m \,|\, F_1(z) \delta_n \rangle|$. Then,

**Lemma 1.** *$S(n)$ is finite for all $n \in \Gamma_1$ and tends to $0$ when $|n| \to \infty$ in $\Gamma_1$.*

*Proof.* By Proposition 2, $S(n) \leqslant \mathit{Const} \sum_{m \in \Gamma_1 \setminus \mathcal{B}(n)} |n - m|^{-\frac{1}{2} + \epsilon}$. Moreover, there exists a $C \geqslant 1$ such that for $\mathcal{B}(m) \neq \mathcal{B}(n)$, $C^{-1}|n - m| \leqslant |n - m'| \leqslant C|n - m|$ for all $m' \in \mathcal{B}(m)$. Since the cardinalities of the $\mathcal{B}(m)$ are bounded, Assumption (A) yields the result. □

For $z \in \mathcal{S}$ let us decompose $F_1(z)$ into two summands: a block-diagonal, $D_1(z) = \sum_{n \in \Gamma_1} \sum_{m \in \mathcal{B}(n)} \langle \delta_n \,|\, F_1(z) \delta_m \rangle \langle \delta_m \,|\, \cdot \rangle \delta_n$, and the other part, $K_1(z) = F_1(z) - D_1(z)$. By Proposition 2, $\langle \delta_n \,|\, F_1(e + \mathrm{i}0) \delta_m \rangle$ exists for $e \in [a, b]$. In particular, letting $F_1(e) := \sum_{m,n \in \Gamma_1} \langle \delta_n \,|\, F_1(e + \mathrm{i}0) \delta_m \rangle \langle \delta_m \,|\, \cdot \rangle \delta_n$, $\lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} F_1(z) = F_1(e)$ weakly. Let us define $D_1(e)$ and $K_1(e)$ in a similar way, so they are weak limits of $D_1(z)$ and $K_1(z)$ respectively.

**Lemma 2.** *For any $e \in [a, b]$, $\lim_{z \to e, z \in \mathbb{C}_+} D_1(z) = D_1(e)$ uniformly.*

*Proof.* Let $\{\mathcal{A}_j\}_{j=1}^L$ be the list of all subsets of $\{m \in \mathbb{Z}^d\ ;\ |m|_1 \leqslant 1\}$ containing 0. For all $n \in \Gamma$ there exists exactly one $j$, which we denote by $\mathrm{j}(n)$, such that $\mathcal{B}(n) - n = \mathcal{A}_j$. Thus, by translational invariance

$$D_1(z) = \sum_{j=1}^{L} \sum_{p,q\in\mathcal{A}_j} \langle \delta_q \,|\, F_1(z)\delta_p\rangle \sum_{\mathrm{j}(n)=j} \langle \delta_{n+p} \,|\, \cdot\rangle \delta_{n+q}.$$

The result follows. □

**Lemma 3.** *Let $\varepsilon > 0$. There exists a finite-dimensional projection $P_\varepsilon$ such that, letting $M_\varepsilon(z) := P_\varepsilon K_1(z) P_\varepsilon$, $\|K_1(z) - M_\varepsilon(z)\| \leqslant \varepsilon$ for all $z \in \overline{\mathcal{S}}$. Moreover, $M_\varepsilon(z)$ is continuous on $\overline{\mathcal{S}}$ (with respect to the uniform operator topology).*

*Proof.* By Lemma 1, there exists a finite set $\mathcal{F} \subset \Gamma_1$ such that for all $z \in \overline{\mathcal{S}}$

$$\sup_{n\in\Gamma_1\setminus\mathcal{F}} \sum_{m\in\Gamma_1} |\langle \delta_n \,|\, K_1(z)\delta_m\rangle| + \sup_{n\in\mathcal{F}} \sum_{m\in\Gamma_1\setminus\mathcal{F}} |\langle \delta_n \,|\, K_1(z)\delta_m\rangle| \leqslant \varepsilon. \tag{2.1}$$

Let $P_\varepsilon$ be the projection onto the vector space generated by $\{\delta_n\}_{n\in\mathcal{F}}$. Notice that $M_\varepsilon(z)$ is weakly continuous and hence uniformly continuous on $\overline{\mathcal{S}}$. Moreover, $\langle \delta_n \,|\, (K_1(z) - M_\varepsilon(z))\delta_m\rangle = \langle \delta_m \,|\, (K_1(z) - M_\varepsilon(z))\delta_n\rangle$ for all $m, n \in \Gamma_1$, so the equation (2.1) is equivalent to $\|K_1(z) - M_\varepsilon(z)\|_1 = \|K_1(z) - M_\varepsilon(z)\|_\infty \leqslant \varepsilon$. Schur's interpolation theorem then completes the proof. □

**Lemma 4.** *For any $e \in [a, b]$, $\lim_{z\to e, z\in\mathbb{C}_+} K_1(z) = K_1(e)$ uniformly.*

*Proof.* Let $\varepsilon > 0$. For $e \in [a, b]$ and $z \in \mathcal{S}$

$$\begin{aligned}\|K_1(z) - K_1(e)\| &\leqslant \|K_1(z) - M_\varepsilon(z)\| + \|K_1(e) - M_\varepsilon(e)\| + \|M_\varepsilon(z) - M_\varepsilon(e)\| \\ &\leqslant \|M_\varepsilon(z) - M_\varepsilon(e)\| + 2\varepsilon.\end{aligned}$$

Since $\lim_{z\to e, z\in\mathbb{C}_+} M_\varepsilon(z) = M_\varepsilon(e)$ uniformly, the proof is complete. □

By Lemmas 2 and 4, $F_1(z)$ has a continuous extension on $\mathbb{C}_+ \cup [a, b]$, so we have reached that $\|F_1(e + \mathrm{i}0)\| < \infty$ for all $e \in [a, b]$. Let us focus on $l^2(\Gamma)$. By the previous work, $F_0(z)$ is continuous on $\overline{\mathcal{S}}$ and analytic in $\mathcal{S}$. Moreover,

**Lemma 5.** *$F_0(z)$ is invertible in $\mathcal{B}(l^2(\Gamma))$ for all $z \in \mathcal{S}$.*

*Proof.* Let $\mu_\varphi$ be the spectral measure of a unit vector $\varphi \in l^2(\Gamma)$ with respect to $\Delta$. For a fixed $z = e + \mathrm{i}y \in \mathcal{S}$, $\Im\langle \varphi \,|\, F_0(z)\varphi\rangle = y \int_{-2d}^{2d} ((t - e)^2 + y^2)^{-1}\, \mathrm{d}\mu_\varphi(t)$. This expression is bounded away from zero when $\varphi$ varies in the unit vectors. Thus, the closure of the numerical range of $F_0(z)$ is included in $\mathbb{C}_+$. The result follows. □

Let $D_0(z) = \mathbf{1}_0 D_1(z) \mathbf{1}_0$ and $K_0(z) = \mathbf{1}_0 K_1(z) \mathbf{1}_0$. By Lemma 3, $K_0(z)$ is a compact operator for any $z \in \overline{\mathcal{S}}$. Moreover, $D_0(z)$ is diagonal; it is indeed a constant (times the identity on $l^2(\Gamma)$) by translational invariance. By Theorem 6.1 in [16], the number $\inf_{z\in\overline{\mathcal{S}}} \Im D_0(z)$, which we denote by $I$, is positive.

**Lemma 6.** *$1 + D_0(z)V$ is invertible in $\mathcal{B}(l^2(\Gamma))$ for any $z \in \overline{\mathcal{S}}$.*

*Proof.* First, $(1 + D_0(z)V)^{-1}$ exists, since $I > 0$. If[2] $|V(n)D_0(z)| \leqslant 1/2$, then $|(1 + D_0(z)V)^{-1}(n)| \leqslant 2$. Otherwise, $|(1 + D_0(z)V)^{-1}(n)| \leqslant 2|D_0(z)|/I$. Hence, $(1 + D_0(z)V)^{-1}$ is bounded, as claimed. □

We now transfer our result from the free resolvent to $P_0$. Since $V$ is bounded, by the argument in Lemma 5, $P_0(z)$ is invertible for each $z \in \mathcal{S}$. Moreover,

**Lemma 7.** *There exists a closed set of Lebesgue measure zero, $\mathcal{R} \subset [a, b]$, such that $P_0$ has a continuous extension $\overline{\mathcal{S}} \setminus \mathcal{R} \longrightarrow \mathcal{B}(l^2(\Gamma))$.*

*Proof.* Let $z \in \mathcal{S}$. By the resolvent identity, $(1 + F_0(z)V)P_0(z) = F_0(z)$. Notice that $1 + F_0(z)V$ is invertible, since $P_0(z)$ and $F_0(z)$ are. Thus,

$$P_0(z) = (1 + F_0(z)V)^{-1}F_0(z), \tag{2.2}$$

where $z \in \mathcal{S}$. One wonders to which extent $(1 + F_0(z)V)^{-1}$ is still invertible when $z \in \partial\mathcal{S}$. Indeed, for any $z \in \overline{\mathcal{S}}$, $1 + F_0(z)V = (1 - K(z))(1 + D_0(z)V)$, where $K(z) := -K_0(z)V(1 + D_0(z)V)^{-1}$ is compact. Since for $z \in \mathcal{S}$ both $1 + D_0(z)V$ and $1 + F_0(z)V$ are invertible, $1 - K(z)$ is. By Proposition 3, $1 - K(z)$ is thus invertible in $\mathcal{B}(l^2(\Gamma))$ for all $z \in [a, b] \setminus \mathcal{R}$, where $\mathcal{R} \subset [a, b]$ is a closed set of Lebesgue measure zero. Hence, the right side in (2.2) extends continuously up to $\overline{\mathcal{S}} \setminus \mathcal{R}$, as desired. □

**Lemma 8.** *There exists a closed set of Lebesgue measure zero, $\mathcal{R} \subset [a, b]$, such that $P_1$ has a continuous extension $\overline{\mathcal{S}} \setminus \mathcal{R} \longrightarrow \mathcal{B}(l^2(\Gamma_1))$.*

*Proof.* By the resolvent identity, $F_1(z)\mathbf{1}_0 - P_1(z)\mathbf{1}_0 = F_1(z)VP_0(z)$. Since $F_1(z)$ and $P_0(z)$ extend continuously up to $\overline{S} \setminus \mathcal{R}$, $P_1(z)\mathbf{1}_0$ also does. By the resolvent identity again, $F_1(z) - P_1(z) = P_1(z)\mathbf{1}_0 V F_1(z)$. The result follows. □

In particular, $\|P_1(e + \mathrm{i}0)\| < \infty$ on $[a, b]$. Since the analogous relation has been established for $F_1$, Proposition 1, the arbitrariness of $[a, b]$, and the absolute continuity of the spectrum of $\Delta$ yield the theorem.

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